

Zariski Topologies for Coprime and Second Submodules*

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Abstract

Let M be a non-zero module over an associative (not necessarily commutative) ring. In this paper, we investigate the so-called *second* and *coprime* submodules of M . Moreover, we topologize the spectrum $\text{Spec}^s(M)$ of second submodules of M and the spectrum $\text{Spec}^c(M)$ of coprime submodules of M , study several properties of these spaces and investigate their interplay with the algebraic properties of M .

1 Introduction

Several papers considered the so called *top modules*, i.e. modules over commutative rings whose spectrum of *prime submodules* attains a Zariski-like topology, e.g. [Lu1995, Lu1997, Lu1999, MMS1997, MMS1998]. In [Abu2006] and [Abu2008], the author investigated and topologized the spectrum of *fully coprime subbicomodules* of a given non-zero duo bicomodule over a coring. Recently, he introduced module theoretic versions of these results in [Abu2011], where a *dual Zariski topology* was introduced on the spectrum of *fully coprime submodules* of a given non-zero duo module over an associative ring. Moreover, he introduced and studied a Zariski topology on the spectrum of *fully prime submodules* of a given non-zero duo module in [Abu].

As a dual notion of *prime submodules*, Yassemi [Yas2001] introduced the notion of *second submodules* of a given non-zero module over a commutative ring. This notion was generalized to modules over arbitrary associative rings by Annin in [Ann2002], where a *second module* was called a *coprime module*. Moreover, the notion of *coprime submodules* was introduced by Kazemifard et al. [KNR]. In this paper, we investigate conditions under

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which the spectrum $\text{Spec}^s(M)$ of second submodules ($\text{Spec}^c(M)$ of coprime submodules) of a given non-zero *module* M over an arbitrary associative – not necessarily commutative – ring R attains a (dual) Zariski topology. We study these spaces and investigate the interplay between the properties of these topologies and the algebraic properties of ${}_R M$.

After this introductory section, we introduce in Section 2 some preliminaries. In particular, we recall some properties and notions of modules that will be needed in the sequel. Section 3 is devoted to a study of the second and coprime submodules of M . In Section 4, we introduce a *dual Zariski topology* on $\text{Spec}^s(M)$ and study its properties. The results obtained are similar to results on the spectrum $\text{Spec}^{fc}(M)$ of *fully coprime submodules* of ${}_R M$ [Abu2011]. In Section 5, we investigate a *Zariski topology* on $\text{Spec}^c(M)$. The results obtained are similar to results on the spectrum $\text{Spec}^{fp}(M)$ of *fully prime submodules* of ${}_R M$ [Abu2011].

Throughout, R is an associative (not necessarily commutative) ring with $1_R \neq 0_R$. With $\text{Max}(R)$ (resp. $\text{Max}({}_R R)$, $\text{Max}(R_R)$) we denote the spectrum of maximal ideals (resp. maximal left ideals, maximal right ideals) of R . On the other hand, we denote by $\text{Min}(R)$ (resp. $\text{Min}({}_R R)$, $\text{Min}(R_R)$) the set of minimal ideals (resp. minimal left ideals, minimal right ideals) of R . Recall that an ideal \mathfrak{p} of R is said to be (*completely*) *prime* iff for any ideal I, J of R (any $a, b \in R$) with $IJ \subseteq \mathfrak{p}$ ($ab \in \mathfrak{p}$), either $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$ ($a \in \mathfrak{p}$ or $b \in \mathfrak{p}$). With $\text{Spec}(R)$ ($\text{CSpec}(R)$) we denote the spectrum of (completely) prime ideals of R . With $\text{Rad}(R) := \bigcap_{\mathfrak{m} \in \text{Max}(R)} \mathfrak{m}$ ($\text{Prad}(R) := \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$) we denote the (*prime*) *radical* of

R . The set of zero-divisors in R is denoted by $Z(R)$ while the group of invertible elements of R is denoted by $U(R)$. Unless otherwise explicitly mentioned, a module will mean a *left* R -module and an ideal is a *two-sided* ideal.

Moreover, we fix an arbitrary left non-zero R -module ${}_R M$ with ring of endomorphisms $S := \text{End}({}_R M)^{op}$ and consider M as an (R, S) -bimodule in the canonical way. We write $K \leq_R M$ ($K \not\leq_R M$) to indicate that K is a (proper) submodule of M and denote with $\pi_K : M \rightarrow M/K$ the canonical surjection. With $\mathbb{P} \subset \mathbb{Z}$ we denote the set of prime positive integers.

2 Preliminaries

For the convenience of the reader, we recall in this section some definitions and properties of modules that will be used in the sequel. Moreover, we illustrate these notions by introducing several examples. For more information, the interested reader may refer to any book in Module Theory (e.g. [Wis1991]).

2.1. We call $L \leq_R M$ *fully invariant* iff L is also an S -submodule. We call ${}_R M$ *duo* iff every R -submodule of M is fully invariant. The ring R is said to be *left duo* (*right duo*) iff every left (right) ideal is two-sided and to be *left quasi-duo* (*right quasi-duo*) iff every maximal left (right) ideal of R is two-sided. Moreover, R is said to be (*quasi-*) *duo* iff R is left and right (quasi-) duo.

Notation. With $\mathcal{L}(M)$ ($\mathcal{L}^{f.i.}(M)$) we denote the lattice of (fully invariant) R -submodules of M and with $\mathcal{I}_r(R)$ (resp. $\mathcal{I}_l(R)$, $\mathcal{I}(R)$) the lattice of right (resp. left, two-sided) ideals.

For subsets $X, Y \subseteq M$ and $Z \subseteq R$ we set

$$\begin{aligned} (X :_Z Y) &:= \{r \in Z \mid ry \in X \text{ for every } y \in Y\}; \\ (X :_Y Z) &:= \{m \in Y \mid rm \in X \text{ for every } r \in Z\}. \end{aligned}$$

In particular, $\text{ann}_R(Y) := (0 :_R Y)$ and $\text{ann}_M(Z) := (0 :_M Z)$. On the other hand, for any non-empty subsets $K \subseteq M$ and $I \subseteq S$ we set $\text{An}(K) := (0 :_S K)$ and $\text{Ke}(I) := (0 :_M I)$. Moreover, we set

$$\mathcal{L}_m(M) := \{IM \mid I \in \mathcal{I}(R)\} \text{ and } \mathcal{L}_c(M) := \{L \leq_R M \mid L = (0 :_M (0 :_R L))\}.$$

Definition 2.2. We say ${}_R M$ is

- self-injective* iff for every $K \leq_R M$, every $f \in \text{Hom}_R(K, M)$ extends to some $\tilde{f} \in S$;
- intrinsically injective* iff $\text{AnKe}(I) = I$ for every finitely generated right ideal I of S ;
- self-cogenerator* iff M cogenerates all its factor R -modules.

2.3. By $\text{Max}(M)$ ($\text{Max}^{\text{f.i.}}(M)$), we denote the *possibly empty* class of maximal R -submodules of M (the class of maximal (R, S) -subbimodules of ${}_R M_S$). For every $L \leq_R M$, we set

$$\mathcal{M}(L) := \{K \in \text{Max}(M) \mid K \supseteq L\} \text{ and } \mathcal{M}^{\text{f.i.}}(L) := \{K \in \text{Max}^{\text{f.i.}}(M) \mid K \supseteq L\}.$$

2.4. Let $L \leq_R M$. We say that L is *superfluous* or *small* in M , and write $L \ll M$, iff $L + \tilde{L} \neq M$ for every $\tilde{L} \not\leq_R M$. The *radical* of M is defined as

$$\text{Rad}(M) := \bigcap_{L \in \text{Max}(M)} L = \sum_{L \ll M} L \quad (:= M \text{ iff } \text{Max}(M) = \emptyset).$$

Definition 2.5. We say ${}_R M$ is

local iff M contains a proper R -submodule that contains every proper R -submodule of M , equivalently iff $\sum_{L \leq_R M} L \neq M$ (this is also equivalent to ${}_R M$ being cyclic, or finitely generated, and having a unique maximal submodule);

hollow (or *couniform*) iff for any $L_1, L_2 \not\leq_R M$ we have $L_1 + L_2 \leq_R M$, equivalently iff every proper R -submodule of M is superfluous;

coatomic (or *B-module* [Fai1976], *Bass module* [Ann2002]) iff every proper R -submodule of M is contained in a maximal R -submodule of M , equivalently iff $\text{Rad}(M/L) \neq M/L$ for every $L \leq_R M$;

f.i.-coatomic iff $\mathcal{M}^{\text{f.i.}}(L) \neq \emptyset$ for every $L \leq_R^{\text{f.i.}} M$, equivalently iff ${}_R M_S$ is coatomic.

2.6. By $\mathcal{S}(M)$ ($\mathcal{S}_{\text{f.i.}}(M)$) we denote the *possibly empty* class of simple R -submodules of M (simple (R, S) -subbimodules of ${}_R M_S$). Let $L \leq_R M$. We set

$$\mathcal{S}(L) := \{K \in \mathcal{S}(M) \mid K \subseteq L\} \text{ and } \hat{\mathcal{S}}(L) := \{K \in \mathcal{S}_{\text{f.i.}}(M) \mid K \subseteq L\}.$$

We say that L is *essential* or *large* in M , and write $L \trianglelefteq M$ iff $L \cap \tilde{L} \neq 0$ for every $0 \neq \tilde{L} \leq_R M$. The *socle* of ${}_R M$ is defined as

$$\text{Soc}(M) := \sum_{L \in \mathcal{S}(M)} L = \bigcap_{L \trianglelefteq M} L \quad (:= 0 \text{ iff } \mathcal{S}(M) = \emptyset)$$

Definition 2.7. We call ${}_R M$:

homogenous semisimple iff M is a (direct) sum of isomorphic simple R -submodules;

completely inhomogenous semisimple iff M is a (direct) sum of pairwise non-isomorphic simple submodules.

Definition 2.8. We say ${}_R M$ is

colocal (or *cocyclic* [Wis1991], *subdirectly irreducible* [AF1974]) iff M contains a *smallest* non-zero R -submodule that is contained in every non-zero R -submodule of M , equivalently

iff $\bigcap_{0 \neq L \leq_R M} L \neq 0$;

uniform iff for any $0 \neq L_1, L_2 \leq_R M$, also $L_1 \cap L_2 \neq 0$, equivalently iff every non-zero R -submodule of M is essential;

atomic iff every $0 \neq L \leq_R M$ contains a simple R -submodule, equivalently iff $\text{Soc}(L) \neq 0$ for every $0 \neq L \leq_R M$;

f.i.-atomic iff $\widehat{S}(L) \neq \emptyset$ for every $0 \neq L \leq_R^{\text{f.i.}} M$, equivalently iff ${}_R M_S$ is atomic.

Example 2.9. ([Wis1991, 17.13]) Let $p \in \mathbb{P}$ and consider the *Prüfer p -group*

$$\mathbb{Z}_{p^\infty} := \sum_{n \in \mathbb{N}} \mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}) = \bigcup_{n \in \mathbb{N}} \mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}) \subseteq \mathbb{Q}/\mathbb{Z}.$$

Every non-zero proper submodule of \mathbb{Z}_{p^∞} is of the form $\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z})$ for some $n \in \mathbb{N}$ (whence finite) and is fully invariant (i.e. \mathbb{Z}_{p^∞} is duo). Clearly, \mathbb{Z}_{p^∞} is Artinian but not Noetherian. Moreover, \mathbb{Z}_{p^∞} is uniserial whence hollow and uniform (i.e. *biuniform* after [CLVW2006]). Notice that \mathbb{Z}_{p^∞} is not local. On the other hand, \mathbb{Z}_{p^∞} is colocal (being an injective hull of \mathbb{Z}_p).

Remark 2.10. Hollow modules are not necessarily coatomic (e.g. \mathbb{Z}_{p^∞} is clearly hollow but not coatomic). This shows that the claim of [Gon1998] about hollow modules being coatomic is not correct.

Examples 2.11. 1. By [Var1979, Proposition 1.14], \mathbb{Z}_{p^∞} and \mathbb{Z}_{p^k} – where p is some prime and $k \in \mathbb{N}$ – are the only hollow Abelian groups (up to isomorphism). Since local modules are precisely the cyclic hollow modules, it follows that the set of local Abelian groups (up to isomorphism) is $\{\mathbb{Z}_{p^k} \mid p \in \mathbb{P} \text{ and } k \in \mathbb{N}\}$. Any colocal Abelian group is either injective ($\simeq \mathbb{Z}_{p^\infty}$ for some $p \in \mathbb{P}$) or finite [Wis1991, 34.14].

2. Examples of uniform modules include, in addition to \mathbb{Z}_{p^∞} and \mathbb{Z}_{p^k} – where $p \in \mathbb{P}$ and $k \in \mathbb{N}$ – any additive subgroup of \mathbb{Q} . Any commutative domain R is obviously uniform when considered as an R -module in the canonical way.

3. Let ${}_R M$ be a non-zero uniserial module (e.g. the Abelian group \mathbb{Z}_{p^∞}). Then ${}_R M$ is trivially hollow and uniform. Moreover, ${}_R M$ is local (colocal) if and only if $\text{Rad}(M) \neq M$ ($\text{Soc}({}_R M) \neq 0$).

4. The Abelian group \mathbb{Z} is uniform but not atomic. On the other hand, \mathbb{Z} is coatomic but not hollow. If $M = S_1 \oplus S_2$, where ${}_R S_1$ and ${}_R S_2$ are simple, then ${}_R M$ is coatomic but not hollow.

Definition 2.12. R is said to be

1. *left Lewy ring* (or *left semiartinian*, or *left socular* [Fai1976, 22.10]) iff every non-zero left R -module has a simple submodule;
2. *left Bass ring* [CLVW2006, 2.19] (or a *B-ring* [Fai1976, 22.7]) iff every non-zero left R -module has a maximal submodule.

Right Lewy rings and right Bass rings can be defined analogously. A left and right Lewy (Bass) rings is said to be a Lewy (Bass) ring.

Remarks 2.13. For any ring R , one can deduce immediately from the definitions:

1. R is a left Lewy ring if and only if every non-zero left R -module has essential socle if and only if every non-zero left R -module is atomic.
2. R is a left Bass ring if and only if every non-zero left R -module has small radical if and only if every non-zero left R -module is coatomic.

Example 2.14. Every right (left) perfect ring is a left Lewy ring [Bac1995] (left Bass ring [CLVW2006]). Any semiprimary ring is a Bass ring.

2.15. ${}_R M$ is said to be *multiplication* iff every $L \leq_R M$ is of the form $L = IM$ for some $I \in \mathcal{I}(R)$, equivalently $L = (L :_R M)M$ (i.e. $\mathcal{L}(M) = \mathcal{L}_m(M)$). It is obvious that every multiplication module is duo. Multiplication modules over commutative rings have been studied intensively in the literature (e.g. [AS2004, PC1995, Smi1994]). Several results in these papers were generalized to multiplication modules over rings *close to be commutative* (e.g. [Tug2003], [Tug2004]).

Recall that a commutative ring R is arithmetical iff $R_{\mathfrak{m}}$ is a chain ring for every $\mathfrak{m} \in \text{Max}(R)$ [Fuc1949].

Example 2.16. A commutative ring R is arithmetical if and only if every finitely generated ideal of R is multiplication. Let R be arithmetical and $I \in \mathcal{I}(R)$ be finitely generated. For every $\mathfrak{m} \in \text{Max}(R)$, the $R_{\mathfrak{m}}$ -ideal $I_{\mathfrak{m}}$ is finitely generated whence principal since $R_{\mathfrak{m}}$ is a chain ring. Moreover, R is an fqp-ring, i.e. ${}_R I$ is self-projective, equivalently $\overline{R}I$ is projective [AJK]. Since I is locally principle and $\overline{R}I$ is projective, it follows by [Smi1994, Theorem A] that ${}_R I$ is multiplication. On the other hand, if every finitely generated ideal of R is multiplication, then every finitely generated ideal of R is locally principal by [Smi1994, Theorem A], whence R is arithmetical by [Jen1966].

2.17. ${}_R M$ is said to be *comultiplication* iff every $L \leq_R M$ is of the form $L = (0 :_M I)$ for some $I \in \mathcal{I}(R)$, equivalently $L = (0 :_M (0 :_R L))$ (i.e. $\mathcal{L}(M) = \mathcal{L}_c(M)$). A ring R for which ${}_R R$ (R_R) is a comultiplication module is called a *left dual* (*right dual*) *ring*. A left dual and right dual ring is said to be a *dual ring*. For more information on comultiplication modules and dual rings, the interested reader is referred to [A-TF2007], [AS] and [NY2003].

Examples 2.18. Let R be left quasi-duo and assume that $PI = IP$ for every $P \in \text{Max}(R)$ and $I \in \mathcal{I}(R)$.

1. All multiplication left R -modules are coatomic. The proof is similar to that for multiplication modules over commutative rings [A-ES1988, Theorem 2.5 (i)] (see also [Zha2006, Proposition 2.4] for the details in the non-commutative case).
2. All comultiplication left R -modules are atomic. The proof is similar to that for comultiplication modules over commutative rings [A-TF2008, Theorem 3.2 (a)].

Notation. For any $L \in \text{Max}(M)$ we set

$$L^e := \bigcap_{K \in \text{Max}(M) \setminus \{L\}} K \quad (:= M \text{ iff } \text{Max}(M) = \{L\}).$$

Dually, for every $L \in \mathcal{S}(M)$ we set

$$L_e := \sum_{K \in \mathcal{S}(M) \setminus \{L\}} K \quad (:= 0 \text{ iff } \mathcal{S}(M) = \{L\}).$$

In [Abu] and [Abu2011], we introduced the class of modules with the (*complete*) *max-property* and the class of modules with the *min-property*. For a study and survey on these modules see [Smi-1]:

2.19. We say that ${}_R M$ has the *complete max-property*, iff for any $L \in \text{Max}(M)$ we have $L^e \not\subseteq L$. We also say that ${}_R M$ has the *max-property*, iff for any $L \in \text{Max}(M)$ and any finite subset $\mathcal{A} \subseteq \text{Max}(M) \setminus \{L\}$ we have $\bigcap_{K \in \mathcal{A}} K \not\subseteq L$.

Lemma 2.20. ([Abu, Lemma 3.15]) *Let ${}_R M$ be self-projective and duo. Then M has the max-property.*

2.21. We say that ${}_R M$ has the *min-property* iff for any simple R -submodule $L \in \mathcal{S}(M)$ we have $L \not\subseteq L_e$. Since simple modules are cyclic, ${}_R M$ has the min-property if and only if for any $L \in \mathcal{S}(M)$ and any finite subset $\{L_1, \dots, L_n\} \subseteq \mathcal{S}(M) \setminus \{L\}$, we have $L \not\subseteq \sum_{i=1}^n L_i$.

Lemma 2.22. ([Abu2011, Lemma 3.17]) *If ${}_R M$ is self-injective and duo, then ${}_R M$ has the min-property.*

Topological Spaces

In what follows, we fix some definitions and notions for topological spaces. For further information, the reader might consult any book in General Topology (e.g. [Bou1966]).

Definition 2.23. We call a topological space \mathbf{X} (*countably*) *compact* iff every open cover of \mathbf{X} has a (countable) finite subcover. Countably compact spaces are also called *Lindelöf spaces*. Note that some authors (e.g. [Bou1966, Bou1998]) assume that compact spaces are in addition Hausdorff.

2.24. We say a topological space \mathbf{X} is *Noetherian* (*Artinian*) iff every ascending (descending) chain of open sets is stationary, equivalently iff every descending (ascending) chain of closed sets is stationary.

Definition 2.25. (e.g. [Bou1966], [Bou1998]) A non-empty topological space \mathbf{X} is called

1. *ultraconnected* iff the intersection of any two non-empty closed subsets is non-empty.
2. *irreducible* (or *hyperconnected*) iff \mathbf{X} is not the union of two proper *closed* subsets, equivalently iff the intersection of any two non-empty open subsets is non-empty.
3. *connected* iff \mathbf{X} is not the *disjoint* union of two proper *closed* subsets, equivalently iff the only subsets of \mathbf{X} that are *clopen* (i.e. closed and open) are \emptyset and \mathbf{X} .

2.26. ([Bou1966], [Bou1998]) Let \mathbf{X} be a non-empty topological space. A non-empty subset $\mathcal{A} \subseteq \mathbf{X}$ is said to be *irreducible* iff it's an irreducible space w.r.t. the relative (subspace) topology. A maximal irreducible subspace of \mathbf{X} is called an *irreducible component*. An irreducible component of a topological space is necessarily closed. Every irreducible subset of \mathbf{X} is contained in an irreducible component of \mathbf{X} , whence \mathbf{X} is the union of its irreducible components. The irreducible components of a Hausdorff space are just the singleton sets.

Lemma 2.27. *The following are equivalent for $\mathcal{A} \subseteq \mathbf{X}$:*

1. \mathcal{A} is irreducible;
2. For any closed subsets $\mathcal{A}_1, \mathcal{A}_2$ of \mathbf{X} :

$$\mathcal{A} \subseteq \mathcal{A}_1 \cup \mathcal{A}_2 \Rightarrow \mathcal{A} \subseteq \mathcal{A}_1 \text{ or } \mathcal{A} \subseteq \mathcal{A}_2. \quad (1)$$

3. For any open subsets $\mathcal{U}_1, \mathcal{U}_2$ of \mathbf{X} :

$$\mathcal{U}_1 \cap \mathcal{A} \neq \emptyset \neq \mathcal{U}_2 \cap \mathcal{A} \Rightarrow (\mathcal{U}_1 \cap \mathcal{U}_2) \cap \mathcal{A} \neq \emptyset. \quad (2)$$

Definition 2.28. Let \mathbf{X} be a topological space and $\mathcal{Y} \subseteq \mathbf{X}$ a closed subset. A point $y \in \mathcal{Y}$ is said to be a *generic point* iff $\mathcal{Y} = \overline{\{y\}}$. If every irreducible closed subset of \mathbf{X} has a unique generic point, then we call \mathbf{X} a *Sober* space.

Definition 2.29. A collection \mathcal{G} of subsets of a topological space \mathbf{X} is *locally finite* iff every point of \mathbf{X} has a neighborhood that intersects only finitely many elements of \mathcal{G} .

3 Coprime and second submodules

As before, ${}_R M$ is a non-zero left module over the associative ring R and $S := \text{End}_R(M)^{op}$. In this section, we introduce and investigate the spectrum $\text{Spec}^s(M)$ of second submodules of M and the spectrum $\text{Spec}^c(M)$ of coprime submodules of M .

Definition 3.1. We call ${}_R M$ (*completely*) *coprime* iff for every $I \in \mathcal{I}(R)$ ($r \in R$) we have $IM = M$ or $IM = 0$ ($rM = M$ or $rM = 0$). Moreover, we say that $K \not\leq_R M$ is (*completely*) *coprime in* M , or a (*completely*) *coprime submodule*, iff for every $I \in \mathcal{I}(R)$ ($r \in R$) we have $IM + K = M$ or $IM \subseteq K$ ($rM + K = M$ or $rM \subseteq K$). On the other hand, we say that $0 \neq L \leq_R M$ is a (*completely*) *second submodule* of M iff ${}_R L$ is a (*completely*) coprime module.

3.2. We set

$$\begin{aligned}\text{Spec}^c(M) &: = \{K \leq_R M \mid K \text{ is coprime in } M\}; \\ \text{Spec}^{cc}(M) &: = \{K \leq_R M \mid K \text{ is completely coprime in } M\}.\end{aligned}$$

We say that ${}_R M$ is *coprimeless* (*c-coprimeless*) iff $\text{Spec}^c(M) = \emptyset$ ($\text{Spec}^{cc}(M) = \emptyset$). On the other hand, we set

$$\begin{aligned}\text{Spec}^s(M) &: = \{0 \neq K \leq_R M \mid K \text{ is a second submodule}\}; \\ \text{Spec}^{cs}(M) &: = \{0 \neq K \leq_R M \mid K \text{ is a completely second submodule}\}.\end{aligned}$$

We say that ${}_R M$ is *secondless* (*c-secondless*) iff $\text{Spec}^s(M) = \emptyset$ ($\text{Spec}^{cs}(M) = \emptyset$).

Example 3.3. ([Smi-2]) A finitely generated non-zero Abelian group G is coprime if and only if G is divisible or homogenous semisimple: Let G be coprime. If $pG = G$ for every $p \in \mathbb{P}$, then G is divisible. If $pG = 0$ for some $p \in \mathbb{P}$, then $G \simeq \bigoplus \mathbb{Z}/p\mathbb{Z}$ is homogenous semisimple. The converse is obvious.

Remarks 3.4. 1. ${}_R M$ is (completely) coprime if and only if 0 is (completely) coprime in M if and only if ${}_R M$ is a (completely) second submodule of itself.

2. Let R be commutative. Then ${}_R M$ is coprime if and only if ${}_R M$ is completely coprime.

Example 3.5. Let R be a simple ring that is not a division ring (e.g. $\mathbb{M}_n(D)$, $n \geq 2$, the matrix ring of $n \times n$ -matrices with entries in some division ring D). Then ${}_R R$ (R_R) is coprime but not completely coprime.

Examples 3.6. 1. The Abelian group \mathbb{Q} is coprime.

2. Let V be a vector space over a division ring D . Then ${}_D V$ is completely coprime, every $W \leq_D V$ is completely coprime in V and every $0 \neq W \leq_D V$ is completely second in V .

3. Every maximal submodule $L \leq_R M$ is coprime in M , i.e. $\text{Max}(M) \subseteq \text{Spec}^c(M)$. In particular, if R is a *left max ring*, i.e. a ring over which every non-zero left R -module has a maximal submodule (e.g. a left perfect ring, or a left V-ring), then $\text{Spec}^c(M) \neq \emptyset$.

4. Every simple submodule $0 \neq L \leq_R M$ is second in M , i.e. $\mathcal{S}(M) \subseteq \text{Spec}^s(M)$. In particular, $\text{Spec}^s(M) \neq \emptyset$.

Example 3.7. Every *homogenous* semi-simple R -module is clearly coprime. Consider the semisimple abelian group $M := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Then ${}_Z M$ is not coprime. Notice that setting $I := 3\mathbb{Z}$ we have $0 \neq IM = \mathbb{Z}/2\mathbb{Z} \neq M$. This shows that the condition homogenous in cannot be removed.

Proposition 3.8. (Compare with [Wij2006, WW2009]) *The following are equivalent:*

1. ${}_R M$ is coprime;
2. $\text{ann}_R(M) = \text{ann}_R(M/L)$ for every $L \leq_R^{\text{f.i.}} M$;

3. $\text{ann}_R(M) = \text{ann}_R(M/L)$ for every $L \leq_R M$;
4. every $L \leq_R M$ is coprime in M ;
5. every $L \leq_R^{\text{f.i.}} M$ is coprime in M ;
6. $R/\text{ann}_R(M)$ is cogenerated by M/L for every $L \leq_R^{\text{f.i.}} M$;
7. $R/\text{ann}_R(M)$ is cogenerated by M/L for every $L \leq_R M$.

Proof. (1) \Rightarrow (2) Let $L \leq_R^{\text{f.i.}} M$ and $I := \text{ann}_R(M/L)$. Suppose that $\text{ann}_R(M) \subsetneq I$. Since $IM \neq 0$, it follows that $IM = M$, a contradiction.

(2) \Rightarrow (3) Let $L \leq_R M$ and $I := \text{ann}_R(M/L)$. Then $I \subseteq \text{ann}_R(M/IM) = \text{ann}_R(M)$ (notice that $IM \leq_R^{\text{f.i.}} M$).

(3) \Rightarrow (4) Let $L \leq_R M$ and $I \in \mathcal{I}(R)$. If $\tilde{L} := IM + L \subsetneq M$, then $I \subseteq \text{ann}_R(M/\tilde{L}) = \text{ann}_R(M)$ whence $IM = 0 \subseteq L$.

(4) \Rightarrow (5) obvious.

(5) \Rightarrow (1) Set $L := 0 \leq_R^{\text{f.i.}} M$.

The equivalences (2) \Leftrightarrow (6) and (3) \Leftrightarrow (7) follow from immediately from the fact that, over any ring, a module N cogenerates the ring if and only if it is faithful over it. ■

Example 3.9. Let ${}_R M$ be *non-Hopf kernel* (i.e. $M/K \simeq M$ for every $K \leq_R M$ [HM1987, HM1987]). Then ${}_R M$ is clearly coprime. In particular, the Prüfer group \mathbb{Z}_{p^∞} is coprime as \mathbb{Z} -module.

The following result follows from the definition and Proposition 3.8.

Proposition 3.10. *The following are equivalent for $K \leq_R M$:*

1. K is (completely) coprime in M ;
2. L is (completely) coprime in M whenever $K \leq_R L \leq_R^{\text{f.i.}} M$;
3. L is (completely) coprime in M whenever $K \leq_R L \leq_R M$;
4. M/L is a (completely) coprime R -module whenever $K \leq_R L \leq_R M$;
5. M/L is a (completely) coprime R -module whenever $K \leq_R L \leq_R^{\text{f.i.}} M$;
6. M/K is a (completely) coprime R -module.

Example 3.11. (cf. [Ann2002, Example 102]) Let R be a UFD with an infinite representative set of the prime elements $\mathbb{P}(R)$ (e.g. $R = \mathbb{Z}$). Consider $M = \prod_{p \in \mathbb{P}(R)} R/pR$ and $K = \bigoplus_{p \in \mathbb{P}} R/pR$. Let $r \in R$ and let $D_r = \{p_1, \dots, p_k\} \subset \mathbb{P}(R)$ be a representative set of the prime divisors of r . Clearly, $rM = \prod_{r \in \mathbb{P}(R) \setminus D_r} R/pR \not\subseteq K$. However, $rM + K = M$. So, K is coprime in M .

Remarks 3.12. 1. If $\text{ann}_R(M) \in \text{Max}(R)$, then ${}_R M$ is coprime: Let $I \in \mathcal{I}(R)$. If $IM \neq 0$, then $\text{ann}_R(M) + I = R$ whence $IM = M$. The converse is not true in general. For example, ${}_Z \mathbb{Q}$ is coprime but $\text{ann}_Z(\mathbb{Q}) = 0$ is not a maximal ideal.

2. If ${}_R M_S$ is simple, then ${}_R M$ is coprime by Proposition 3.8.
3. R is a simple ring if and only if ${}_R R$ (R_R) is coprime. In particular, a commutative ring R is a field if and only if ${}_R R$ is coprime.
4. The following are equivalent:
 - (a) R is a division ring;
 - (b) ${}_R R$ is completely coprime;
 - (c) R_R is completely coprime;
 - (d) ${}_R R$ is coprime and R is (left) duo;
 - (e) R_R is coprime and R is (right) duo.

Proposition 3.13. *Let $0 \neq L \leqslant_R M$. If $K \leqslant_R M$ is (completely) coprime in M , then $K \cap L$ is (completely) coprime in L or $(K + L)/L$ is (completely) coprime in M/L .*

Proof. Let $0 \neq L \leqslant_R M$ and assume that K is (completely) coprime in M .

Case I. $K + L \subsetneq M$. It follows by Proposition 3.10 that $K + L \leqslant_R M$ is (completely) coprime in M whence $\frac{M/L}{(K+L)/L} \simeq M/(K+L)$ is a (completely) coprime R -module and so $(K + L)/L$ is (completely) coprime in M/L .

Case II. $K + L = M$. Consider $K \cap L \subsetneq L$.

For any $I \in \mathcal{I}(R)$ ($r \in R$) we have either $IM + K = M$ ($rM + K = M$) whence

$$\begin{aligned} IL + (K \cap L) &= (IL + K) \cap L = (I(L + K) + K) \cap L = (IM + K) \cap L = M \cap L = L \\ (rL + (K \cap L)) &= (rL + K) \cap L = (r(L + K) + K) \cap L = (rM + K) \cap L = M \cap L = L \end{aligned}$$

or $IM \subseteq K$ ($rM \subseteq K$) whence $IL \subseteq L \cap K$ ($rL \subseteq K \cap L$). Consequently, $K \cap L$ is (completely) coprime in L . ■

Corollary 3.14. *Let $0 \neq L \leqslant_R M$. If L and M/L are coprimeless (c -coprimeless), then ${}_R M$ is also coprimeless (c -coprimeless).*

Lemma 3.15. 1. *If ${}_R M$ is (completely) coprime, then $\text{ann}_R(M)$ is (completely) prime.*

2. *If $K \leqslant_R M$ is (completely) coprime in M , then $\mathfrak{p} := (K :_R M)$ is (completely) prime.*

Proof. 1. Assume that ${}_R M$ is (completely) coprime. Let $IJ \subseteq \text{ann}_R(M)$ ($ab \in \text{ann}_R(M)$) and suppose that $J \not\subseteq \text{ann}_R(M)$ ($b \notin \text{ann}_R(M)$) i.e. $JM \neq 0$ ($bM \neq 0$). Since ${}_R M$ is (completely) coprime, we conclude that $JM = M$ ($bM = M$) whence $IM = I(JM) = (IJ)M = 0$ ($aM = a(bM) = (ab)M = 0$) i.e. $I \subseteq \text{ann}_R(M)$ ($a \in \text{ann}_R(M)$).

2. The result follows from “1” and Proposition 3.10. ■

Definition 3.16. Let $K \leq_R M$ be (completely) coprime and consider the (completely) prime ideal $\mathfrak{p} := (K :_R M)$. We say K is a (completely) \mathfrak{p} -coprime submodule of M .

The following result extends some results in [A-TF2007] to comultiplication modules over non-commutative rings and improves some other results.

Proposition 3.17. 1. Let $_R M$ be multiplication. Then $_R M$ is coprime if and only if $_R M$ is simple.
2. Let $_R M$ be comultiplication. Then $_R M$ is coprime if and only if $\text{ann}_R(M)$ is a prime ideal.

Proof. 1. Clearly, every simple module is coprime. Conversely, let $_R M$ be multiplication and coprime. If $N \leq_R M$, then setting $I := (N :_R M)$ we have $IM \neq M$ whence $N = IM = 0$, i.e. $_R M$ is simple.

2. If $_R M$ is coprime, then $\text{ann}_R(M)$ is a prime ideal by Lemma 3.15. Let $_R M$ be comultiplication and assume that $\text{ann}_R(M) \in \text{Spec}(R)$. Let $I \in \mathcal{I}(R)$ be such that $IM \neq 0$ and let $J := (0 :_R IM)$. Since $JI \subseteq \text{ann}_R(M)$ and $I \not\subseteq \text{ann}_R(M)$ we obtain $J \subseteq \text{ann}_R(M)$ whence $M = (0 :_M J) = (0 :_M (0 :_R IM)) = IM$. Consequently, $_R M$ is coprime. ■

Corollary 3.18. Let $_R M$ be multiplication and comultiplication. The following are equivalent:

1. $_R M$ is coprime;
2. $\text{ann}_R(M)$ is a prime ideal;
3. $_R M$ is simple.

In particular, if R is a prime ring and $_R M$ is faithful, multiplication and comultiplication, then $_R M$ is coprime if and only if $_R M$ is simple.

3.19. Recall that the ring R is said to be *zero-dimensional* iff every prime ideal of R is maximal. Examples of zero-dimensional rings include *biregular rings* [Wis1991, 3.18 (6, 7)] and left (right) perfect rings. For left (right) duo rings, the notion of zero-dimensionality coincides with that of π -regularity [Hir1978]. A prime ring (e.g. a commutative integral domain) is said to be *one-dimensional* iff every non-zero prime ideal is maximal. In particular, commutative Dedekind domains are one-dimensional.

Corollary 3.20. 1. If $_R M$ is multiplication, then $\text{Spec}^c(M) = \text{Max}(M)$.

2. If $_R M$ is comultiplication, then

$$\text{Spec}^s(M) = \{0 \neq L \leq_R M \mid (0 :_R L) \in \text{Spec}(R)\}. \quad (3)$$

If, moreover, R is zero-dimensional, then

$$\text{Spec}^s(M) = \mathcal{S}(M). \quad (4)$$

Proof. In light of Proposition 3.17 we need to prove only the last part of the second statement. Notice that $\mathcal{S}(M) \subseteq \text{Spec}^s(M)$. Assume that every prime ideal of R is maximal. Let $K \in \text{Spec}^s(M)$ so that $(0 :_R K) \in \text{Spec}(R)$ by Lemma 3.15, whence a maximal ideal by our assumption on R . It follows that $K = (0 :_M (0 :_R K))$ is simple: if $0 \neq K_1 \leq_R K$, then $(0 :_R K) \subsetneq (0 :_R K_1) \subsetneq R$, a contradiction. ■

Corollary 3.21. 1. If R is a left due ring (e.g. a commutative ring), then

$$\text{Spec}^c({}_R R) = \text{Max}(R). \quad (5)$$

2. If R is a left dual ring, then

$$\text{Spec}^s({}_R R) = \{I \in \mathcal{I}_l(R) \mid (0 :_R I) \in \text{Spec}(R)\}. \quad (6)$$

If moreover R is zero-dimensional, then

$$\text{Spec}^s({}_R R) = \text{Min}({}_R R). \quad (7)$$

Example 3.22. A ring R is Quasi-Frobenius if and only if R is dual and Artinian. Examples of Quasi-Frobenius rings include semisimple Artinian rings, the group algebra $\mathbb{F}[G]$ where \mathbb{F} is a field and G is a finite group, and R/aR where R is a commutative PID and $0 \neq a \notin U(R)$ (e.g. $\mathbb{Z}/n\mathbb{Z}$, $n \geq 2$).

Example 3.23. Let $B \subseteq \mathbb{Q}$ be the subring consisting of all rational numbers with odd denominators and $M := \mathbb{Q}/B$. Consider the *idealization* $R := B \oplus M$ with multiplication

$$(b_1, m_1)(b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1). \quad (8)$$

Then R is a dual ring which is not Quasi-Frobenius.

Remark 3.24. For any module M , the so-called *generalized associated prime* ideals of ${}_R M$ were introduced in [D-AT2000] as the set

$$\mathbf{Ass}_R(M) := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} = (0 :_R L) \text{ for some } L \leq_R M\}. \quad (9)$$

If ${}_R M$ is comultiplication, then one can easily see that there is a 1-1 correspondence

$$\text{Spec}^s(M) \longleftrightarrow \mathbf{Ass}_R(M), \quad L \mapsto (0 :_R L) \quad (10)$$

with inverse $\mathfrak{p} \mapsto (0 :_M \mathfrak{p})$. In particular, if R is a left dual ring, then there is a 1-1 correspondence

$$\text{Spec}^s({}_R R) \longleftrightarrow \mathbf{Ass}_R(R). \quad (11)$$

3.25. For every $L \leq_R M$ we define

$$\mathcal{V}^s(L) := \{K \in \text{Spec}^s(M) \mid K \subseteq L\} \text{ and } \mathcal{X}^s(L) := \{K \in \text{Spec}^s(M) \mid K \not\subseteq L\}.$$

For every $\mathcal{A} \subseteq \text{Spec}^s(M)$ we set

$$\mathcal{H}(\mathcal{A}) := \sum_{K \in \mathcal{A}} K \quad (:= 0 \text{ iff } \mathcal{A} = \emptyset). \quad (12)$$

In particular, we set

$$\text{Corad}_M^s(L) := \mathcal{H}(\mathcal{V}^s(L)) = \sum_{K \in \mathcal{V}^s(L)} K \quad (:= 0 \text{ iff } \mathcal{V}^s(L) = \emptyset) \quad (13)$$

and

$$\text{Corad}^s(M) := \text{Corad}_M^s(M) = \sum_{K \in \text{Spec}^s(M)} K \quad (:= 0 \text{ iff } \text{Spec}^s(M) = \emptyset). \quad (14)$$

We say $L \leq_R M$ is *s-coradical in M* iff $\text{Corad}_M^s(L) = L$. In particular, we call ${}_R M$ an *s-coradical module* iff $\text{Corad}^s(M) = M$.

Remarks 3.26. 1. For any $L \leq_R M$ we have

$$\text{Spec}^s(L) = \mathcal{L}(L) \cap \text{Spec}^s(M) = \mathcal{V}^s(L).$$

2. For any $L_1 \leq_R L_2 \leq_R M$ we have $\text{Corad}_M^s(L_1) \subseteq \text{Corad}_M^s(L_2)$. Moreover, for any $L \leq_R M$ we have

$$\text{Corad}_M^s(\text{Corad}_M^s(L)) = \text{Corad}_M^s(L).$$

3. Notice that $\mathcal{S}(M) \subseteq \text{Spec}^s(M)$. In particular, if ${}_R M$ is atomic, then for every $0 \neq L \leq_R M$ we have $\emptyset \neq \mathcal{S}(L) \subseteq \text{Spec}^s(L) = \mathcal{V}^s(L) \subseteq \text{Spec}^s(M)$.

Definition 3.27. Let $0 \neq L \leq_R M$. A maximal element of $\mathcal{V}^s(L)$, if any, is said to be *maximal under L*. A *maximal element of $\text{Spec}^s(M)$* is said to be a *maximal second submodule of M*.

Lemma 3.28. Let ${}_R M$ be atomic and comultiplication. For every $0 \neq L \leq_R M$ there exists $K \in \text{Spec}^s(M)$ which is maximal under L .

Proof. Let $0 \neq L \leq_R M$. Since ${}_R M$ is atomic, $\emptyset \neq \mathcal{S}(L) \subseteq \mathcal{V}^s(L)$. Let

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq K_{n+1} \subseteq \cdots$$

be an ascending chain in $\mathcal{V}^s(L)$ and set $\tilde{K} := \bigcup_{i=1}^{\infty} K_i$. Then we have a descending chain of prime ideals

$$(0 :_R K_1) \supseteq (0 :_R K_2) \supseteq \cdots \supseteq (0 :_R K_n) \supseteq (0 :_R K_{n+1}) \supseteq \cdots \quad (15)$$

and it follows that $\mathfrak{p} := (0 :_R \tilde{K}) = \bigcap_{i=1}^{\infty} (0 :_R K_i)$ is a prime ideal. Since ${}_R M$ is comultiplication, $\tilde{K} \in \mathcal{V}^s(L)$ by Corollary 3.20. By Zorn's Lemma, $\mathcal{V}^s(L)$ has a maximal element. ■

3.29. For every $L \leq_R M$ we define

$$\mathcal{V}^c(L) := \{K \in \text{Spec}^c(M) \mid L \subseteq K\} \text{ and } \mathcal{X}^c(L) := \{K \in \text{Spec}^c(M) \mid L \not\subseteq K\}.$$

For every $\mathcal{A} \subseteq \text{Spec}^s(M)$, we set

$$\mathcal{J}(\mathcal{A}) := \bigcap_{K \in \mathcal{A}} K \quad (:= M \text{ iff } \mathcal{A} = \emptyset). \quad (16)$$

and

$$\text{Rad}_M^c(L) := \mathcal{J}(\mathcal{V}^c(L)) = \bigcap_{K \in \mathcal{V}^c(L)} K \quad (:= M \text{ iff } \mathcal{V}^c(L) = \emptyset). \quad (17)$$

In particular,

$$\text{Rad}^c(M) := \text{Rad}_M^c(0) = \bigcap_{K \in \text{Spec}^c(M)} K \quad (:= M \text{ iff } \text{Spec}^c(M) = \emptyset). \quad (18)$$

We say $L \leq_R M$ is *c-radical* iff $\text{Rad}_M^c(L) = L$.

Remarks 3.30. 1. For any $L \leq_R M$ let

$$\mathcal{U}(L) := \{\tilde{L} \leq_R M \mid L \leq_R \tilde{L} \leq_R M\}. \quad (19)$$

We have a bijection

$$\text{Spec}^c(M) \cap \mathcal{U}(L) \leftrightarrow \text{Spec}^c(M/L).$$

2. For any $L_1 \leq_R L_2 \leq_R M$ we have $\text{Rad}_M^c(L_1) \subseteq \text{Rad}_M^c(L_2)$. Moreover, for any $L \leq_R M$ we have

$$\text{Rad}_M^c(\text{Rad}_M^c(L)) = \text{Rad}_M^c(L).$$

3. $\text{Max}(M) \subseteq \text{Spec}^c(M)$. If ${}_R M$ is coatomic, then for every $L \leq_R M$ we have $\emptyset \neq \mathcal{M}(L) \subseteq \mathcal{V}^c(L) \subseteq \text{Spec}^c(M)$.

Definition 3.31. We call ${}_R M$ (*completely*) *endo-coprime* iff M_S is (completely) coprime. Moreover, we say that $K \leq_{\pm R}^{\text{f.i.}} M$ is a (*completely*) *endo-coprime* *R-submodule* iff $K \leq_S M$ is a (completely) coprime submodule.

Proposition 3.32. 1. If ${}_R M$ is (*completely*) *endo-coprime*, then S is a prime ring (a domain).

2. Let M_S be duo and ${}_R M$ be a self-cogenerator. Then ${}_R M$ is *endo-coprime* if and only if S is a prime ring.

Proof. 1. If ${}_R M$ is (*completely*) *endo-coprime*, then – by definition – M_S is (completely) coprime and it follows by Lemma 3.15 that $0 = \text{ann}_S(M)$ is a (completely) prime ideal, i.e. S is a prime ring (a domain).

2. Assume that M_S is duo and ${}_R M$ is a self-cogenerator. Let $L \leq_S M$ be an arbitrary submodule. Since M_S is duo, $L \leq_B M$ where $B := \text{End}(M_S)$. Considering the canonical ring morphism $\beta : R \rightarrow B$, we know that M is a (B, S) -bimodule and conclude that $L \leq_R^{\text{f.i.}} M$ whence $(0 :_S L) \in \mathcal{I}(S)$. Since ${}_R M$ is a self-cogenerator, $L = (0 :_M (0 :_S L))$ [Wis1991, 28.1., 28.2.]. Consequently, M_S is a comultiplication module and it follows by Lemma 3.17 that M_S is coprime, i.e. ${}_R M$ is *endo-coprime*. ■

3.33. We say ${}_R M$ is *divisible* iff $rM = M$ for every $r \in R \setminus Z(R)$. The sum of all divisible submodules of ${}_R M$ is a divisible submodule, denoted by $\text{div}(M)$. If $\text{div}(M) = 0$, then ${}_R M$ is said to be *reduced*. Moreover, $L \leq_R M$ is said to be *relatively divisible* iff

$$rL = rM \cap L \text{ for every } r \in R.$$

Proposition 3.34. *Let $K \not\leq_R L \leq_R M$.*

1. *Let ${}_R M$ be flat.*

- (a) *If $L \leq_R M$ is pure and K is coprime in M , then K is coprime in L .*
- (b) *${}_R M$ is coprime if and only if every non-zero pure submodule of M is second in M .*
- (a) *If $L \leq_R M$ is relatively divisible and K is completely coprime in M , then K is completely coprime in L .*
- (b) *${}_R M$ is completely coprime if and only if every non-zero relatively divisible R -submodule of M is completely second in M .*

Proof. 1. Let ${}_R M$ be flat.

- (a) Assume that K is coprime in M . Since ${}_R M$ is flat and $L \leq_R M$ is pure, we have $IL = IM \cap L$ for every $I \in \mathcal{I}(R)$ (e.g. [Wis1991, 36.6]). If $IL \not\subseteq K$ for some $I \in \mathcal{I}(R)$, then indeed $IM \not\subseteq K$ and so

$$IL + K = (IM \cap L) + K = (IM + K) \cap L = M \cap L = L. \quad (20)$$

- (b) The proof is similar to that of (a).

- (a) Assume that K is coprime in M . Since $L \leq M$ is relatively divisible, for every $r \in R$ we have

$$rL = rM \cap L \subseteq K \cap L = K \quad (21)$$

or

$$rL + K = (rM \cap L) + K = (rM + K) \cap L = M \cap L = L.$$

- (b) The proof is similar to that of (a). ■

Lemma 3.35. *Let $K \not\leq_R M$ and $\mathfrak{q} := (K :_R M)$. Then K is completely coprime in M if and only if \mathfrak{q} is completely prime and M/K is a divisible R/\mathfrak{q} -module. In particular, ${}_R M$ is completely coprime if and only if $\overline{R} := R/\text{ann}_R(M)$ is a domain and $\overline{R}M$ is divisible.*

Proof. (\Rightarrow) Assume that K is completely coprime in M , so that \mathfrak{q} is completely prime by Lemma 3.15 and $\overline{R} := R/\mathfrak{q}$ is a domain. Let $m + K \in M/K$ and consider any $0 \neq \bar{r} \in R/\mathfrak{q}$. Since $r \notin \mathfrak{q}$, we have $rM + K = M$ and so $\bar{r}(M/K) = M/K$. Consequently, M/K is a divisible R/\mathfrak{q} -module.

(\Leftarrow) Assume that \mathfrak{q} is completely prime and that M/K is a divisible R/\mathfrak{q} -module. Let $r \in R$ be such that $rM \not\subseteq K$, i.e. $r \notin \mathfrak{q}$. Since M/K is a divisible R/\mathfrak{q} -module, we conclude that $\bar{r}(M/K) = M/K$ and so $rM + K = M$. Consequently, K is completely coprime in M . ■

Corollary 3.36. *If $K \not\leq_R M$ and $(K :_R M) \in \text{Max}({}_R R)$, then K is completely coprime in M . In particular, if $\mathfrak{m} \in \mathcal{I}(R) \cap \text{Max}({}_R R)$ and $\mathfrak{m}M \neq M$, then $\mathfrak{m}M$ is completely coprime in M .*

3.37. ([MR1987], [Mar1972]) Recall that the ring R is said to be *right (left) bounded* iff every essential right (left) ideal of R contains a non-zero *two-sided* ideal. We say that R is bounded iff R is left and right bounded. Let R be a Noetherian prime ring with simple Artinian classical ring of quotients Q . If each ideal of R distinct from zero is invertible in Q , then R is called a *Dedekind prime ring*. For example, all commutative Dedekind domains and full matrix rings over them are bounded Dedekind prime rings. Moreover, all (hereditary Noetherian) prime principal ideal rings are (bounded) Dedekind prime rings.

Lemma 3.38. ([Mar1972, Theorem 3.19]) *Let R be a bounded Dedekind prime ring. Any R -module M possesses a unique largest divisible submodule D such that $M = D \oplus K$ where K has no divisible submodules.*

Definition 3.39. Let R be a bounded Dedekind prime ring, M an R -module and consider the decomposition $M = D \oplus K$ from the previous lemma. The submodule $\text{div}(M) := D$ ($\text{Red}(M) := K$) is called the *divisible (reduced) part* of M .

Proposition 3.40. *Let R be a bounded Dedekind prime ring.*

1. *If ${}_R M$ is completely coprime, then ${}_R M$ is divisible or reduced.*
2. *Let R be a domain. Then $\text{Red}(M)$ is completely coprime in M if and only if ${}_R M$ is not reduced.*

Proof. 1. Let ${}_R M$ be completely coprime. If ${}_R M$ is divisible, then we are done. Suppose that $rM \neq M$ for some $r \in R \setminus Z(R)$. Since ${}_R M$ is completely coprime, we have

$$0 = rM = r(\text{div}(M) \oplus \text{Red}(M)) = \text{div}(M) \oplus r\text{Red}(M).$$

It follows that $\text{div}(M) = 0$, i.e. ${}_R M$ is reduced.

2. Let R be domain. If $\text{Red}(M)$ is completely coprime in M , then in particular $\text{Red}(M) \leqslant_R M$ and so ${}_R M$ is not reduced (i.e. ${}_R M$ is divisible or mixed). On the other hand, assume that $\text{Red}(M) \leqslant_R M$. Since R is a domain, we conclude that

$$(\text{Red}(M) :_R M) = (0 :_R \text{div}(M)) = 0.$$

Applying Lemma 3.35, we conclude that $\text{Red}(M)$ is completely coprime in M . ■

Corollary 3.41. 1. *If ${}_R M$ is c -coprimeless, then $\mathfrak{m}M = M$ for every $\mathfrak{m} \in \mathcal{I}(R) \cap \text{Max}({}_R R)$.*

2. *Let R be a bounded Dedekind prime domain. If ${}_R M$ is c -coprimeless, then ${}_R M$ is reduced.*

Proposition 3.42. *Let $\{M_\lambda\}_\Lambda$ be a family of non-zero R -modules.*

1. *We have:*

- (a) *If $\prod_{\lambda \in \Lambda} M_\lambda$ is (completely) coprime, then ${}_R M_\lambda$ is (completely) coprime for every $\lambda \in \Lambda$.*

(b) If $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is (completely) coprime, then ${}_R M_\lambda$ is (completely) coprime for every $\lambda \in \Lambda$.

2. Assume that $\text{ann}_R(M_\lambda) = \mathfrak{p} = \text{ann}_R(M_\gamma)$ for every $\lambda, \gamma \in \Lambda$.

(a) $\prod_{\lambda \in \Lambda} M_\lambda$ is (completely) \mathfrak{p} -coprime if and only if ${}_R M_\lambda$ is (completely) \mathfrak{p} -coprime for every $\lambda \in \Lambda$.

(b) $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is (completely) \mathfrak{p} -coprime if and only if ${}_R M_\lambda$ is (completely) \mathfrak{p} -coprime for every $\lambda \in \Lambda$.

Proof. Notice that $\prod_{\lambda \in \Lambda} M_\lambda \neq 0 \neq \bigoplus_{\lambda \in \Lambda} M_\lambda$.

1. (a) Assume that $M := \prod_{\lambda \in \Lambda} M_\lambda$ is (completely) coprime. For any $\lambda \in \Lambda$ and any $I \in \mathcal{I}(R)$ ($r \in R$): If $IM_\lambda \neq M_\lambda$ ($rM_\lambda \neq M_\lambda$), then $IM \neq M$ ($rM \neq M$) and so $IM = 0$ ($rM = 0$). Whence $IM_\lambda = 0$ ($rM_\lambda = 0$) for every $\lambda \in \Lambda$.

(b) The proof is similar to that of (a).

2. Since $\text{ann}_R(M_\lambda) = \mathfrak{p} = \text{ann}_R(M_\gamma)$ for every $\lambda, \gamma \in \Lambda$, we have $\text{ann}_R(\prod_{\lambda \in \Lambda} M_\lambda) = \mathfrak{p} = \text{ann}_R(\bigoplus_{\lambda \in \Lambda} M_\lambda)$.

(a) Let $M := \prod_{\lambda \in \Lambda} M_\lambda$. Assume that ${}_R M_\lambda$ is (completely) \mathfrak{p} -coprime for every $\lambda \in \Lambda$. For any $I \in \mathcal{I}(R)$ ($r \in R$): If $IM \neq M$ ($rM \neq M$), then $IM_{\lambda_0} \neq M_{\lambda_0}$ ($rM_{\lambda_0} \neq M_{\lambda_0}$) for some $\lambda_0 \in \Lambda$. Since ${}_R M_\lambda$ is (completely) \mathfrak{p} -coprime, $IM_{\lambda_0} = 0$ ($rM_{\lambda_0} = 0$) whence $I \subseteq \mathfrak{p}$ ($r \in \mathfrak{p}$) i.e. $IM = 0$ ($rM = 0$). It follows that ${}_R M$ is (completely) \mathfrak{p} -coprime.

(b) The proof is similar to that of (a). ■

Proposition 3.43. If $\{M_\lambda\}_\Lambda$ is a family of coprimeless (c-coprimeless) R -modules, then $\prod_{\lambda \in \Lambda} M_\lambda$ and $\bigoplus_{\lambda \in \Lambda} M_\lambda$ are coprimeless (c-coprimeless).

Proof. Let $\{M_\lambda\}_\Lambda$ be a family of coprimeless (c-coprimeless) R -modules and set $M := \prod_{\lambda \in \Lambda} M_\lambda$. Suppose that $K \in \text{Spec}^c(M)$ ($K \in \text{Spec}^{cc}(M)$) so that, in particular, $\pi_{\lambda_0}(K) \subsetneq M_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. For every $I \in \mathcal{I}(R)$ ($r \in R$) we have $IM \subseteq K$ ($rM \subseteq K$) whence $I\pi_{\lambda_0}(M) \subseteq \pi_{\lambda_0}(K)$ ($r\pi_{\lambda_0}(M) \subseteq \pi_{\lambda_0}(K)$) or $IM + K = M$ ($rM + K = M$) whence $IM_{\lambda_0} + \pi_{\lambda_0}(K) = M_{\lambda_0}$ ($rM_{\lambda_0} + \pi_{\lambda_0}(K) = M_{\lambda_0}$). It follows that $\pi_{\lambda_0}(K) \in \text{Spec}^c(M_{\lambda_0})$ ($\pi_{\lambda_0}(K) \in \text{Spec}^{cc}(M_{\lambda_0})$), a contradiction. ■

The ring R is said to be *binoetherian* (or *weakly Noetherian* [Row2008, page 74]) iff R satisfies the ACC on $\mathcal{I}(R)$.

Example 3.44. (cf. [Ann2002, Proposition 107]) Let R be a binoetherian ring. Then $\mathfrak{q}M$ is coprime in M for some prime ideal $\mathfrak{q} \in \text{Spec}(R)$. In particular, $\text{Spec}^c(M) \neq \emptyset$.

Proof. Since R is binoetherian, for every $J \in \mathcal{I}(R)$ there exist (e.g. [Row2008, Theorem 16.24], [Row2006, Theorem 9.2]) $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \text{Spec}(R)$ such that

$$\mathfrak{q}_1 \bullet \cdots \bullet \mathfrak{q}_k \subseteq J \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k. \quad (22)$$

In particular, there exist $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \text{Spec}(R)$ such that $\mathfrak{p}_1 \bullet \cdots \bullet \mathfrak{p}_n = 0$, whence

$$\mathcal{E} := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p}M \neq M\} \neq \emptyset.$$

By assumption, \mathcal{E} has a maximal element \mathfrak{q} . Let $I \in \mathcal{I}(R)$.

Case I: $I \subseteq \mathfrak{q}$. In this case, $IM \subseteq \mathfrak{q}M$.

Case II: $I \not\subseteq \mathfrak{q}$, so that $\mathfrak{q} \subsetneq J := I + \mathfrak{q}$.

By (22), there exist $\mathfrak{q}_1, \dots, \mathfrak{q}_k \in \text{Spec}(R)$ such that

$$\mathfrak{q}_1 \bullet \cdots \bullet \mathfrak{q}_k \subseteq J \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k.$$

Since \mathfrak{q} is maximal in \mathcal{E} , we have $\mathfrak{q}_j M = M$ for all $j = 1, \dots, k$, whence

$$IM + \mathfrak{q}M = JM = M.$$

Consequently, $\mathfrak{q}M \in \text{Spec}^c(M)$. ■

4 Top^s-modules

As before M is a non-zero left R -module. In this section we topologize the spectrum of second submodules of ${}_R M$ and investigate the properties of the induced topology. Several proofs in this section are similar to proofs of results in [Abu2011], whence omitted.

Notation. Set

$$\begin{aligned} \xi^s(M) &:= \{\mathcal{V}^s(L) \mid L \in \mathcal{L}(M)\}; & \xi_c^s(M) &:= \{\mathcal{V}^s(L) \mid L \in \mathcal{L}_c(M)\}; \\ \tau^s(M) &:= \{\mathcal{X}^s(L) \mid L \in \mathcal{L}(M)\}; & \tau_c^s(M) &:= \{\mathcal{X}^s(L) \mid L \in \mathcal{L}_c(M)\}; \\ \mathbf{Z}^s(M) &:= (\text{Spec}^s(M), \tau^s(M)); & \mathbf{Z}_c^s(M) &:= (\text{Spec}^s(M), \tau_c^s(M)). \end{aligned}$$

Lemma 4.1. *Consider the class of varieties $\xi^s(M)$.*

1. $\mathcal{V}^s(0) = \emptyset$ and $\mathcal{V}^s(M) = \text{Spec}^s(M)$;
2. $\bigcap_{\lambda \in \Lambda} \mathcal{V}^s(L_\lambda) = \mathcal{V}^s(\bigcap_{\lambda \in \Lambda} L_\lambda)$ for any $\{L_\lambda\}_\Lambda \subseteq \mathcal{L}(M)$;
3. For any $I, \tilde{I} \in \mathcal{I}(R)$, we have

$$\mathcal{V}^s((0 :_M I)) \cup \mathcal{V}^s((0 :_M \tilde{I})) = \mathcal{V}^s((0 :_M I) + (0 :_M \tilde{I})) = \mathcal{V}^s((0 :_M I \cap \tilde{I})) = \mathcal{V}^s((0 :_M I\tilde{I})). \quad (23)$$

Proof. Statements “1”, “2” and the inclusions

$$\mathcal{V}^s((0 :_M I)) \cup \mathcal{V}^s((0 :_M \tilde{I})) \subseteq \mathcal{V}^s((0 :_M I) + (0 :_M \tilde{I})) \subseteq \mathcal{V}^s((0 :_M I \cap \tilde{I})) \subseteq \mathcal{V}^s((0 :_M I\tilde{I}))$$

in (3) are clear. Let $K \in \mathcal{V}^s((0 :_M I\tilde{I}))$ and suppose that $K \not\subseteq (0 :_M \tilde{I})$, whence $\tilde{I}K = K$. It follows that $IK = I(\tilde{I}K) = (I\tilde{I})K = 0$, i.e. $K \subseteq (0 :_M I)$. Consequently, $K \in \mathcal{V}^s((0 :_M I)) \cup \mathcal{V}^s((0 :_M \tilde{I}))$. ■

4.2. We call $0 \neq L \leq_R M$

strongly hollow in M iff for any $L_1, L_2 \leq_R M$ we have

$$L \subseteq L_1 + L_2 \Rightarrow L \subseteq L_1 \text{ or } L \subseteq L_2; \quad (24)$$

completely hollow, iff for any collections $\{L_\lambda\}_\Lambda$ of R -submodules of M we have:

$$L = \sum L_\lambda \Rightarrow L = L_\lambda \text{ for some } \lambda \in \Lambda; \quad (25)$$

Remark 4.3. Strongly hollow submodules were considered briefly in [RRW2005] under the name *\vee -coprime submodules*. Completely hollow modules were introduced under the name *completely coirreducible modules* in [A-TF2008]; however, the zero submodule was allowed to be completely coirreducible which does not fit with our scheme.

Notation. We set

$$\mathcal{SH}(M) := \{L \leq_R M \mid L \text{ is strongly hollow in } M\}. \quad (26)$$

Remarks 4.4. 1. If ${}_R M$ is uniserial, then every submodule of M is strongly hollow.

2. If $\mathcal{S}(M) \subseteq \mathcal{SH}(M)$, then ${}_R M$ has the min-property.

Example 4.5. Let M be an n -dimensional vector space over a division ring D . If $n \geq 2$, then M has a vector subspace which is hollow but not strongly hollow: Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V and consider $L = D(v_1 + \dots + v_n)$. Then L is clearly hollow, being 1-dimensional, but not strongly hollow in M since $L \subseteq Dv_1 + \dots + Dv_n$ but $L \not\subseteq Dv_i$ for any $i = 1, \dots, n$. In particular, $\{(x, y) \mid y = x\}$ is hollow but not strongly hollow in \mathbb{R}^2 . If M is a uniserial non-zero module with $0 \neq L \leq_R M$ not finitely generated, then clearly L is strongly hollow but not completely hollow. In particular, the Abelian group \mathbb{Z}_{p^∞} is strongly hollow but not completely hollow.

In general, $\xi^s(M)$ is not closed under finite unions. This motivates

Definition 4.6. We call ${}_R M$ a *top^s-module* iff $\xi^s(M)$ is closed under finite unions, equivalently iff $\mathbf{Z}^s(M) := (\text{Spec}^s(M), \tau^s(M))$ is a topological space.

Remark 4.7. If ${}_R M$ is secondless (i.e. $\text{Spec}^s(M) = \emptyset$), then ${}_R M$ is trivially a top^s-module.

Theorem 4.8. 1. $\mathbf{Z}_c^s(M) := (\text{Spec}^s(M), \tau_c^s(M))$ is a topological space.

2. If $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$, then ${}_R M$ is a top^s-module.

Proof. 1. This follows directly from Lemma 4.1.

2. This follows from the observation that $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$ if and only if $\mathcal{V}^s(L_1) \cup \mathcal{V}^s(L_2) = \mathcal{V}^s(L_1 + L_2)$ (equivalently, $\mathcal{X}^s(L_1 + L_2) = \mathcal{X}^s(L_1) \cap \mathcal{X}^s(L_2)$) for any $L_1, L_2 \leq_R M$. ■

Proposition 4.9. *Let ${}_R M$ be comultiplication.*

1. *Every second submodule of M is strongly hollow (i.e. $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$).*
2. *Every finitely generated second submodule of M is completely hollow.*
3. *${}_R M$ is a top^s -module.*
4. *${}_R M$ has the min-property.*

Proof. Let ${}_R M$ be comultiplication.

1. This follows directly from Lemma 4.1 and the definition of comultiplication modules.
2. This follow directly from the definitions and “1”.
3. This follows from “1” and Theorem 4.8.
4. This follows from “1”, which yields $\mathcal{S}(M) \subseteq \text{Spec}^s(M) \subseteq \mathcal{SH}(M)$. ■

Lemma 4.10. *Let ${}_R M$ be a top^s -module. The closure of any subset $\mathcal{A} \subseteq \text{Spec}^s(M)$ is*

$$\overline{\mathcal{A}} = \mathcal{V}^s(\mathcal{H}(\mathcal{A})). \quad (27)$$

Remarks 4.11. Let ${}_R M$ be a top^s -module and consider the Zariski topology

$$\mathbf{Z}^s(M) := (\text{Spec}^s(M), \tau^s(M)). \quad (28)$$

1. $\mathbf{Z}^s(M)$ is a T_0 (Kolmogorov) space.
2. Let $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$ (e.g. ${}_R M$ is comultiplication). Then

$$\mathcal{B} := \{\mathcal{X}^s(L) \mid L \leq_R M \text{ is finitely generated}\} \quad (29)$$

is a basis of open sets for $\mathbf{Z}^s(M)$.

3. If $L \in \text{Spec}^s(M)$, then $\overline{\{L\}} = \mathcal{V}^s(L)$. In particular, for any $K \in \text{Spec}^s(M)$:

$$K \in \overline{\{L\}} \Leftrightarrow K \subseteq L.$$

4. $\mathcal{X}^s(L) = \emptyset \Rightarrow \text{Soc}(M) \subseteq L$. The converse holds if, for example, $\mathcal{S}(M) = \text{Spec}^s(M)$.
5. Let ${}_R M$ be atomic. For every $L \leq_R M$ we have $\mathcal{V}^s(L) = \emptyset$ if and only if $L = 0$.

6. Let $0 \neq L \leq_R M$. The embedding

$$\iota : \text{Spec}^s(L) \rightarrow \text{Spec}^s(M)$$

induces as continuous map

$$\iota : \mathbf{Z}^s(L) \rightarrow \mathbf{Z}^s(M).$$

This follows from the fact that $\iota^{-1}(\mathcal{V}^s(N)) = \mathcal{V}^s(N \cap L)$ for every R -submodule $N \leq_R M$.

Notation. Set

$$\mathbf{CL}(\mathbf{Z}^s(M)) := \{\mathcal{A} \subseteq \text{Spec}^s(M) \mid \mathcal{A} = \overline{\mathcal{A}}\} \text{ and } \mathcal{CR}^s(M) := \{L \leq_R M \mid \text{Corad}_M^s(L) = L\}.$$

Theorem 4.12. *Let ${}_R M$ be a top^s -module.*

1. *We have an order-preserving bijection*

$$\mathcal{CR}^s(M) \longleftrightarrow \mathbf{CL}(\mathbf{Z}^s(M)), \quad L \mapsto \mathcal{V}^s(L). \quad (30)$$

2. *$\mathbf{Z}^s(M)$ is Noetherian if and only if ${}_R M$ satisfies the DCC condition on $\mathcal{CR}^s(M)$.*

3. *$\mathbf{Z}^s(M)$ is Artinian if and only if ${}_R M$ satisfies the ACC condition on $\mathcal{CR}^s(M)$.*

Theorem 4.13. *Let ${}_R M$ be a top^s -module. If ${}_R M$ is Artinian (Noetherian), then $\mathbf{Z}^s(M)$ is Noetherian (Artinian).*

Proposition 4.14. *Let ${}_R M$ be a top^s -module and $\mathcal{A} \subseteq \text{Spec}^s(M)$.*

1. *If $\mathcal{A} \subseteq \text{Spec}^s(M)$ is irreducible, then $\mathcal{H}(\mathcal{A})$ is a second submodule of M .*

2. *Let $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$. The following are equivalent:*

- (a) *$\mathcal{A} \subseteq \text{Spec}^s(M)$ is irreducible;*
- (b) *$\mathcal{H}(\mathcal{A})$ is a second submodule of M ;*
- (c) *$0 \neq \mathcal{H}(\mathcal{A}) \leq_R M$ is strongly hollow.*

Proof. 1. Assume that \mathcal{A} is irreducible. By definition, $\mathcal{A} \neq \emptyset$ and so $\mathcal{H}(\mathcal{A}) \neq 0$. Let $I \in \mathcal{I}(R)$ and suppose that $I\mathcal{H}(\mathcal{A}) \neq \mathcal{H}(\mathcal{A})$ and $I\mathcal{H}(\mathcal{A}) \neq 0$. Set $\mathcal{A}_1 := \{K \in \mathcal{A} \mid IK = K\}$ and $\mathcal{A}_2 := \{K \in \mathcal{A} \mid IK = 0\}$. Then $\mathcal{A} \subseteq \mathcal{V}^s(\mathcal{H}(\mathcal{A}_1)) \cup \mathcal{V}^s(\mathcal{H}(\mathcal{A}_2))$. Notice that $\mathcal{A} \not\subseteq \mathcal{V}^s(\mathcal{H}(\mathcal{A}_1))$ (otherwise, $I\mathcal{H}(\mathcal{A}) = I\mathcal{H}(\mathcal{A}_1) = \mathcal{H}(\mathcal{A}_1) = \mathcal{H}(\mathcal{A})$) and $\mathcal{A} \not\subseteq \mathcal{V}^s(\mathcal{H}(\mathcal{A}_2))$ (otherwise, $I\mathcal{H}(\mathcal{A}) = I\mathcal{H}(\mathcal{A}_2) = 0$), a contradiction. Consequently, $I\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{A})$ or $I\mathcal{H}(\mathcal{A}) = 0$, i.e. $\mathcal{H}(\mathcal{A})$ is a second submodule of M .

2. Let $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$.

(a) \Rightarrow (b) follows from “1”.

(b) \Rightarrow (c) is obvious.

(c) \Rightarrow (a) Since $\mathcal{H}(\mathcal{A}) \neq 0$, we conclude that $\mathcal{A} \neq \emptyset$. Suppose that $\mathcal{A} \subseteq \mathcal{V}^s(L_1) \cup \mathcal{V}^s(L_2) \subseteq \mathcal{V}^s(L_1 + L_2)$, i.e. $\mathcal{H}(\mathcal{A}) \subseteq L_1 + L_2$. Since $\mathcal{H}(\mathcal{A})$ is strongly hollow, we conclude that $\mathcal{H}(\mathcal{A}) \subseteq L_1$ whence $\mathcal{A} \subseteq \mathcal{V}^s(L_1)$ or $\mathcal{H}(\mathcal{A}) \subseteq L_2$ whence $\mathcal{A} \subseteq \mathcal{V}^s(L_2)$. Consequently, \mathcal{A} is irreducible. ■

Theorem 4.15. *Let ${}_R M$ be a top^s -module.*

1. (a) *If $\text{Spec}^s(M)$ is irreducible, then $\text{Corad}_M^s(M)$ is a second submodule of M .*
 (b) *If $\mathcal{S}(M)$ is irreducible, then $\text{Soc}(M)$ is a second submodule of M .*
2. *Let $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$.*
 - (a) *The following are equivalent:*
 - i. $\text{Spec}^s(M)$ is irreducible;
 - ii. $\text{Corad}_M^s(M)$ is a second submodule of M ;
 - iii. $0 \neq \text{Corad}_M^s(M) \leq_R M$ is strongly hollow.
 - (b) *The following are equivalent:*
 - i. $\mathcal{S}(M)$ is irreducible;
 - ii. $\text{Soc}(M)$ is a second submodule of M ;
 - iii. $0 \neq \text{Soc}(M) \leq_R M$ is strongly hollow.

Example 4.16. Let ${}_R M$ be a top^s -module. If $\emptyset \neq \mathcal{A} \subseteq \text{Spec}^s(M)$ is a chain, then \mathcal{A} is irreducible. In particular, if ${}_R M$ is uniserial, then $\text{Spec}^s(M)$ is irreducible.

Notation. Set

$$\text{Max}(\text{Spec}^s(M)) := \{K \in \text{Spec}^s(M) \mid K \text{ is a maximal second submodule of } M\}. \quad (31)$$

Proposition 4.17. *Let $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$ (whence ${}_R M$ be a top^s -module).*

1. *The bijection (30) restricts to a bijection:*

$$\text{Spec}^s(M) \longleftrightarrow \{\mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^s(M) \text{ is an irreducible closed subset}\} \quad (32)$$

2. *The bijection (32) restricts to a bijection*

$$\text{Max}(\text{Spec}^s(M)) \longleftrightarrow \{\mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^s(M) \text{ is an irreducible component}\}.$$

Proof. Recall the bijection $\mathcal{CR}^s(M) \xrightarrow{\mathcal{V}^s(-)} \mathbf{CL}(\mathbf{Z}^s(M))$.

1. Let $K \in \text{Spec}^s(M)$. Then $K = \mathcal{H}(\mathcal{V}^s(K))$ and so the closed set $\mathcal{V}^s(K)$ is irreducible by Proposition 4.14 “2”. On the other hand, let $\mathcal{A} \subseteq \text{Spec}^s(M)$ be a closed irreducible subset. Notice that $\mathcal{H}(\mathcal{A})$ is second in M by Proposition 4.14 “2” and that $\mathcal{A} = \overline{\mathcal{A}} = \mathcal{V}^s(\mathcal{H}(\mathcal{A}))$.
2. Let K be maximal in $\text{Spec}^s(M)$. Then $\mathcal{V}^s(K)$ is irreducible by “1”. Let \mathcal{Y} be the irreducible component containing $\mathcal{V}^s(K)$. Since \mathcal{Y} is closed, $\mathcal{Y} = \mathcal{V}^s(L)$ for some $L \in \text{Spec}^s(M)$. Since $\mathcal{V}^s(K) \subseteq \mathcal{V}^s(L)$ we have $K \subseteq L$. Since $K \in \text{Max}(\text{Spec}^s(M))$, we conclude that $K = L$ and so $\mathcal{V}^s(K)$ is an irreducible component of $\text{Spec}^s(M)$.

Conversely, let \mathcal{Y} be an irreducible component of $\text{Spec}^s(M)$. Since \mathcal{Y} is closed and irreducible, it follows by “1” that $\mathcal{Y} = \mathcal{V}^s(L)$ for some $L \in \text{Spec}^s(M)$. Suppose that L is not maximal in $\text{Spec}^s(M)$, i.e. there exists $K \in \text{Spec}^s(M)$ such that $L \subsetneq K \subseteq M$. It follows that $\mathcal{V}^s(L) \subsetneq \mathcal{V}^s(K)$, a contradiction since $\mathcal{V}^s(K) \subseteq \text{Spec}^s(M)$ is irreducible by “1”. We conclude that L is maximal in $\text{Spec}^s(M)$. ■

Corollary 4.18. *If $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$, then $\text{Spec}^s(M)$ is a Sober space.*

Proof. Let $\mathcal{A} \subseteq \text{Spec}^s(M)$ be an irreducible closed subset. By Proposition 4.17 “1”, $\mathcal{A} = \mathcal{V}^s(K)$ for some $K \in \text{Spec}^s(M)$. It follows that

$$\mathcal{A} = \overline{\mathcal{A}} = \mathcal{V}^s(\mathcal{H}(\mathcal{A})) = \mathcal{V}^s(K) = \overline{\{K\}},$$

i.e. K is a generic point for \mathcal{A} . If L is a generic point of \mathcal{A} , then $\mathcal{V}^s(K) = \mathcal{V}^s(L)$ whence $K = L$. ■

Proposition 4.19. *Let ${}_R M$ be an atomic top^s -module. Then ${}_R M$ is uniform if and only if $\text{Spec}^s(M)$ is ultraconnected.*

Theorem 4.20. *Let ${}_R M$ be an atomic top^s -module.*

1. *If $\mathcal{S}(M)$ is countable, then $\mathbf{Z}^s(M)$ is countably compact.*
2. *If $\mathcal{S}(M)$ is finite, then $\mathbf{Z}^s(M)$ is compact.*

Example 4.21. The Prüfer group $\mathbb{Z}(p^\infty)$ is an atomic top^s -module over \mathbb{Z} . Since $\mathcal{S}(\mathbb{Z}(p^\infty)) = \{\mathbb{Z}(\frac{1}{p} + \mathbb{Q}/\mathbb{Z})\}$ is finite, we conclude that $\mathbf{Z}^s(\mathbb{Z}(p^\infty))$ is compact.

Proposition 4.22. *Let ${}_R M$ be a top^s -module and assume that every second submodule of M is simple.*

1. *If ${}_R M$ has the min-property, then $\text{Spec}^s(M)$ is discrete.*
2. *M has a unique simple R -submodule if and only if ${}_R M$ has the min-property and $\text{Spec}^s(M)$ is connected.*

Proof. 1. If ${}_R M$ has the min-property, then for every $K \in \text{Spec}^s(M) = \mathcal{S}(M)$ we have $\{K\} = \mathcal{X}(\{K\}_e)$ an open set. Since every singleton set is open, $\text{Spec}^s(M)$ is discrete.

2. (\Rightarrow) Assume that ${}_R M$ has a unique simple R -submodule. Clearly, ${}_R M$ has the min-property and $\text{Spec}^s(M)$ is connected since it consists of only one point.

(\Leftarrow) Assume that ${}_R M$ has the min-property and that $\text{Spec}^s(M)$ is connected. By “1”, $\text{Spec}^s(M)$ is discrete and so $\mathcal{S}(M) = \text{Spec}^s(M)$ has only one point since a discrete connected space cannot contain more than one-point. ■

Theorem 4.23. *Let ${}_R M$ be an atomic top^s -module and assume that every second submodule of M is simple. If ${}_R M$ has the min-property, then*

1. *$\text{Spec}^s(M)$ is countably compact if and only if $\mathcal{S}(M)$ is countable.*
2. *$\text{Spec}^s(M)$ is compact if and only if $\mathcal{S}(M)$ is finite.*

As a direct consequence of Proposition 4.22 we obtain:

Theorem 4.24. *Let ${}_R M$ be atomic and assume that $\mathcal{S}(M) = \text{Spec}^s(M) \subseteq \mathcal{SH}(M)$ so that M is a top^s -module. Then ${}_R M$ is colocal if and only if $\text{Spec}^s(M)$ is connected.*

Lemma 4.25. Let $\text{Spec}^s(M) \subseteq \mathcal{SH}(M)$. If $n \geq 2$ and $\mathcal{A} = \{K_1, \dots, K_n\} \subseteq \text{Spec}^s(M)$ is a connected subset, then for every $i \in \{1, \dots, n\}$, there exists $j \in \{1, \dots, n\} \setminus \{i\}$ such that $K_i \leq_R K_j$ or $K_j \leq_R K_i$.

Proposition 4.26. Let ${}_R M$ be an atomic top^s -module with the min-property and let $\emptyset \neq \mathcal{A} = \{K_\lambda\}_\Lambda \subseteq \mathcal{S}(M)$. If $|\mathcal{S}(L)| < \infty$ for every $L \in \text{Spec}^s(M)$, then \mathcal{A} is locally finite.

Lemma 4.27. Let ${}_R M$ be an atomic top^s -module. Then the following are equivalent for any $L \leq_R M$:

1. $L \in \mathcal{S}(M)$;
2. L is a second submodule of M and $\mathcal{V}^s(L) = \{L\}$;
3. $\{L\}$ is closed in \mathbf{Z}_M^s .

Proposition 4.28. If ${}_R M$ is an atomic top^s -module, then $\text{Spec}^s(M) = \mathcal{S}(M)$ if and only if $\mathbf{Z}^s(M)$ is T_1 (Frécht space).

Combining Propositions 4.22 and 4.28 we obtain

Theorem 4.29. Let ${}_R M$ be an atomic top^s -module with the min-property. The following are equivalent:

1. $\text{Spec}^s(M) = \mathcal{S}(M)$;
2. $\mathbf{Z}^s(M)$ is discrete;
3. $\mathbf{Z}^s(M)$ is T_2 (Hausdorff space);
4. $\mathbf{Z}^s(M)$ is T_1 (Frécht space).

5 Top^c -modules

As before, ${}_R M$ is a non-zero left R -module. In this section we topologize the spectrum of coprime submodules of ${}_R M$ and investigate the properties of the induced topology. Several proofs in this section are similar to proofs of similar results in [Abu] and dual results in Section 4, whence omitted.

Notation. Recall that for every $L \leq_R M$ we define

$$\mathcal{V}^c(L) := \{K \in \text{Spec}^c(M) \mid L \subseteq K\} \text{ and } \mathcal{X}^c(L) := \{K \in \text{Spec}^c(M) \mid L \not\subseteq K\}.$$

Moreover, we set

$$\begin{aligned} \xi^c(M) &:= \{\mathcal{V}^c(L) \mid L \in \mathcal{L}(M)\}; & \xi_m^c(M) &:= \{\mathcal{V}^c(L) \mid L \in \mathcal{L}_m(M)\}; \\ \tau^c(M) &:= \{\mathcal{X}^c(L) \mid L \in \mathcal{L}(M)\}; & \tau_m^c(M) &:= \{\mathcal{X}^c(L) \mid L \in \mathcal{L}_m(M)\}; \\ \mathbf{Z}^c(M) &:= (\text{Spec}^c(M), \tau^c(M)); & \mathbf{Z}_m^c(M) &:= (\text{Spec}^c(M), \tau_m^c(M)). \end{aligned}$$

Lemma 5.1. Consider the class of varieties $\xi^c(M)$.

1. $\mathcal{V}^c(M) = \emptyset$ and $\mathcal{V}^c(0) = \text{Spec}^c(M)$;
2. $\bigcap_{\lambda \in \Lambda} \mathcal{V}^c(L_\lambda) = \mathcal{V}^c(\sum_{\lambda \in \Lambda} L_\lambda)$ for any $\{L_\lambda\}_\Lambda \subseteq \mathcal{L}(M)$;
3. For any $I, \tilde{I} \in \mathcal{I}(R)$, we have

$$\mathcal{V}^c(IM) \cup \mathcal{V}^c(\tilde{I}M) = \mathcal{V}^c(IM \cap \tilde{I}M) = \mathcal{V}^c((I \cap \tilde{I})M) = \mathcal{V}^c(I\tilde{I}M).$$

Proof. Statements “1”, “2” and the inclusions

$$\mathcal{V}^c(IM) \cup \mathcal{V}^c(\tilde{I}M) \subseteq \mathcal{V}^c(IM \cap \tilde{I}M) \subseteq \mathcal{V}^c((I \cap \tilde{I})M) \subseteq \mathcal{V}^c(I\tilde{I}M) \quad (33)$$

in (3) are clear. Let $K \in \mathcal{V}^c(I\tilde{I}M)$ and suppose that $K \notin \mathcal{V}^c(\tilde{I}M)$. Then $M = \tilde{I}M + K$, whence $IM = I(\tilde{I}M + K) = I(\tilde{I}M) + IK = (I\tilde{I})M + IK \subseteq K$, i.e. $K \in \mathcal{V}^c(IM)$. ■

In general, $\xi^c(M)$ is not closed under finite unions. This motivates

Definition 5.2. We call ${}_R M$ a *top^c-module* iff $\xi^c(M)$ is closed under finite unions, equivalently $\mathbf{Z}^c(M) := (\text{Spec}^c(M), \tau^c(M))$ is a topological space.

Remark 5.3. If ${}_R M$ is coprimeless, i.e. $\text{Spec}^c({}_R M) = \emptyset$, then ${}_R M$ is trivially a *top^c-module*.

Example 5.4. (cf. [Ann2002, Remark 105]) Let R have a unique prime ideal \mathfrak{p} . For any ${}_R M$ we have

$$\text{Spec}^c({}_R M) = \begin{cases} \mathfrak{p}M, & \mathfrak{p}M \neq M \\ \emptyset, & \mathfrak{p}M = M \end{cases} \quad (34)$$

So, if $\mathfrak{p}M = M$, then M is a *top^c-module*.

Example 5.5. ([Ann2002, Example 106]) Let $R = \mathbb{Q}[x_1, x_2, \dots]$, where $x_i^2 = 0$ for every $i \in \mathbb{N}$, be the commutative local ring with *unique prime ideal* $\mathfrak{p} = (x_1, x_2, \dots)$. Let $E = \{e_1, e_2, \dots\}$ be a countably infinite set, F the free R -module with basis E and M the R -module F modulo the relations $e_i x_i = e_{i-1}$ for each $i \geq 2$. Indeed $\mathfrak{p}M = M$, whence $\text{Spec}^c(M) = \emptyset$. So, M is a *top^c-module*.

Example 5.6. ([Smi-2]) Let R have a unique (completely) prime ideal \mathfrak{m} . If \mathfrak{m} is idempotent, then \mathfrak{m} is (c-coprimeless) coprimeless: Suppose that \mathfrak{m} contains a left subideal $J \subsetneq \mathfrak{m}$ that is (completely) coprime in \mathfrak{m} . By Lemma 3.35, $\mathfrak{p} := (J :_R \mathfrak{m})$ is a (completely) prime ideal, whence $\mathfrak{p} = \mathfrak{m}$ and consequently $\mathfrak{m} = \mathfrak{m}^2 = \mathfrak{p}\mathfrak{m} \subseteq J$, a contradiction. Consequently, \mathfrak{m} is (c-coprimeless) coprimeless. We conclude that if \mathfrak{m} is idempotent, then ${}_R \mathfrak{m}$ is a *top^c-module*.

Example 5.7. Let $R := \mathbb{F}[\mathbb{G}]$, where \mathbb{F} is a field, $G = \mathbb{Z}(p^\infty)$ and $\mathfrak{m} = \sum_{g \in G} R(g-1)$: Notice that $R/\mathfrak{m} \simeq \mathbb{F}$ whence $\mathfrak{m} \in \text{Max}(R)$. For every $x \in G$, there exists $y \in G$ such that $x = y^p$. So $(x-1) = (y-1)^p \in I^p \subseteq \mathfrak{m}^2$. Therefore, $\mathfrak{m} = \mathfrak{m}^2$ and so \mathfrak{m} is the only prime ideal of R . Consequently, ${}_R \mathfrak{m}$ is a *top^c-module*.

5.8. We call $L \leq_R M$:

irreducible iff for any $L_1, L_2 \leq_R M$:

$$L_1 \cap L_2 = L \Rightarrow L_1 = L \text{ or } L_2 = L; \quad (35)$$

strongly irreducible iff for any $L_1, L_2 \leq_R M$:

$$L_1 \cap L_2 \subseteq L \Rightarrow L_1 \subseteq L \text{ or } L_2 \subseteq L. \quad (36)$$

completely irreducible, iff any collections $\{L_\lambda\}_\Lambda$ of R -submodules of ${}_R M$ we have:

$$\bigcap_{\lambda \in \Lambda} L_\lambda = L \Rightarrow L_\lambda = L \text{ for some } \lambda \in \Lambda; \quad (37)$$

Remark 5.9. Strongly irreducible ideals were introduced first by Bourbaki [Bou1998, p. 301, Exercise 34] and named *quasi-prime ideals*. Recently, they were investigated by Heinzer et al. [HRR2002] while the class of completely irreducible ideals was introduced by Fuchs et al. in [FHO2006]. The corresponding notions for submodules of modules over commutative rings were investigated in [E-A2005] and [A-TF2008], respectively.

Notation. We set

$$\mathcal{SI}(M) := \{L \leq_R M \mid L \text{ is strongly irreducible}\}. \quad (38)$$

Examples 5.10. 1. If ${}_R M$ is uniserial, then every submodule of M is strongly irreducible.

2. $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$ if and only if $\mathcal{V}^c(L_1) \cup \mathcal{V}^c(L_2) = \mathcal{V}^c(L_1 \cap L_2)$ (equivalently, $\mathcal{X}^c(L_1) \cap \mathcal{X}^c(L_2) = \mathcal{X}^c(L_1 \cap L_2)$) for all $L_1, L_2 \leq_R M$.

3. If $\text{Max}(M) \subseteq \mathcal{SI}(M)$, then ${}_R M$ has the max-property.

As a direct consequence of Lemma 5.1 we obtain

Theorem 5.11. 1. $\mathbf{Z}_m^c(M) := (\text{Spec}^c(M), \tau_m^c(M))$ is a topological space.

2. If $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$, then ${}_R M$ is a top^c module.

Recall that a left R -module N is said to be *finitely cogenerated* iff for any monomorphism $N \xrightarrow{f} \prod_{\lambda \in \Lambda} N_\lambda$, there exists a finite subset $\{\lambda_1, \dots, \lambda_n\} \subseteq \Lambda$ such that

$$N \xrightarrow{f} \prod_{\lambda \in \Lambda} N_\lambda \xrightarrow{\pi} \bigoplus_{i=1}^n N_{\lambda_i}$$

is injective.

Proposition 5.12. Let ${}_R M$ be a multiplication module.

1. Every coprime submodule of M is strongly irreducible (i.e. $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$).

2. If $L \in \text{Spec}^c(M)$ is such that M/L is finitely cogenerated, then L is completely irreducible.

3. ${}_R M$ is a top^c module.

4. ${}_R M$ has the max -property.

Proof. 1. This follows directly from Lemma 5.1.

2. This follows directly from the definitions and “1”.

3. This follows directly from “1” and Theorem 5.11.

4. This follows directly from “1”, which yields $\text{Max}(M) \subseteq \text{Spec}^c(M) \subseteq \mathcal{SI}(M)$. ■

The following example provides a top^c -module which is not multiplication.

Example 5.13. ([Smi-2]) Consider the Abelian group $G := \bigoplus_{p \in \mathbb{P}} \mathbb{Z}/p\mathbb{Z}$. Let $K \not\leq_R G$ be coprime in G . Notice that $\mathfrak{p} := (K : G) \neq 0$ (otherwise, G/K would be a divisible \mathbb{Z} -module by Lemma 3.35, a contradiction). So, $\mathfrak{p} = q\mathbb{Z}$ for some $q \in \mathbb{P}$ and $K = \bigoplus_{p \in \mathbb{P} \setminus \{q\}} \mathbb{Z}/p\mathbb{Z}$. On the other hand, for every $q \in \mathbb{P}$, the \mathbb{Z} -submodule $K := qG$ is coprime in G . Consequently,

$$\text{Spec}^c(G) = \{qG \mid q \in \mathbb{P}\} \text{ and } \text{Rad}^c(G) = \bigcap_{q \in \mathbb{P}} qG = 0.$$

Since G is semisimple, every subgroup $L \leq_{\mathbb{Z}} G$ is of the form $L = \sum_{p \in \Lambda} \mathbb{Z}/p\mathbb{Z}$ for some

$\Lambda \subseteq \mathbb{P}$. Let $qG \in \text{Spec}^c(G)$ be arbitrary and suppose there are subsets $\Lambda, \tilde{\Lambda} \subseteq \mathbb{P}$ such that $\left(\sum_{p \in \Lambda} \mathbb{Z}/p\mathbb{Z}\right) \cap \left(\sum_{p \in \tilde{\Lambda}} \mathbb{Z}/p\mathbb{Z}\right) \subseteq qG$. Then indeed $q \in \Lambda \cup \tilde{\Lambda}$ and it follows that $\sum_{p \in \Lambda} \mathbb{Z}/p\mathbb{Z} \subseteq qG$ or $\sum_{p \in \tilde{\Lambda}} \mathbb{Z}/p\mathbb{Z} \subseteq qG$. We conclude that every coprime subgroup of G is strongly irreducible, i.e. $\text{Spec}^c({}_{\mathbb{Z}}G) \subseteq \mathcal{SI}({}_{\mathbb{Z}}G)$, whence ${}_{\mathbb{Z}}G$ is a top^c -module by Theorem 5.11. Notice that ${}_{\mathbb{Z}}G$ is not a multiplication module: Let $\Lambda \subseteq \mathbb{P}$ be a finite subset and $N := \bigoplus_{p \in \Lambda} \mathbb{Z}/p\mathbb{Z}$. Then $(N :_{\mathbb{Z}} G) = \bigcap_{p \in \mathbb{P} \setminus \Lambda} p\mathbb{Z} = 0$, whence $N \neq (N :_{\mathbb{Z}} G)G$ and ${}_{\mathbb{Z}}G$ is not a multiplication module.

Lemma 5.14. *Let ${}_R M$ be a top^c -module. The closure of any $\mathcal{A} \subseteq \text{Spec}^c(M)$ is*

$$\overline{\mathcal{A}} = \mathcal{V}^c(\mathcal{J}(\mathcal{A})). \quad (39)$$

Remarks 5.15. Let ${}_R M$ be a top^c -module and consider the Zariski topology

$$\mathbf{Z}^c(M) := (\text{Spec}^c(M), \tau^c(M)). \quad (40)$$

1. $\mathbf{Z}^c(M)$ is a T_0 (Kolmogorov) space.

2. Set $\mathcal{X}_m^c := \mathcal{X}^c(Rm)$ for each $m \in M$. The set

$$\mathcal{B} := \{\mathcal{X}_m^c \mid m \in M\}$$

is a basis of open sets for the Zariski topology $\mathbf{Z}^c(M)$.

3. If $L \in \text{Spec}^c(M)$, then $\overline{\{L\}} = \mathcal{V}^c(L)$. In particular, for any $K \in \text{Spec}^c(M)$:

$$K \in \overline{\{L\}} \Leftrightarrow L \subseteq K.$$

4. $\mathcal{X}^c(L) = \emptyset \Rightarrow L \subseteq \text{Rad}(M)$. The converse holds if, for example, $\text{Max}(M) = \text{Spec}^c(M)$.

5. Let ${}_R M$ be coatomic. For every $L \leq_R M$ we have $\mathcal{V}^c(L) = \emptyset$ if and only if $L = M$.

6. Let $L \leq_R M$. The mapping

$$\pi : \text{Spec}^c(M) \rightarrow \text{Spec}^c(M/L), \quad N \mapsto (N + L)/L$$

induces a continuous mapping

$$\pi : \mathbf{Z}^c(M) \longrightarrow \mathbf{Z}^c(M/L). \quad (41)$$

This follows from the fact that $\pi^{-1}(\mathcal{V}^c(N/L)) = \mathcal{V}^c(N)$ for every $L \leq_R N \leq_R M$.

Notation. Set

$$\mathbf{CL}(\mathbf{Z}^c(M)) := \{\mathcal{A} \subseteq \text{Spec}^c(M) \mid \mathcal{A} = \overline{\mathcal{A}}\} \text{ and } \mathcal{R}^c(M) := \{L \leq_R M \mid \text{Rad}_M^c(L) = L\}.$$

Theorem 5.16. *Let ${}_R M$ be a top^c -module.*

1. *We have an order-reversing bijection*

$$\mathcal{R}^c(M) \longleftrightarrow \mathbf{CL}(\mathbf{Z}^c(M)), \quad L \mapsto \mathcal{V}^c(L). \quad (42)$$

2. *$\mathbf{Z}^c(M)$ is Noetherian if and only if ${}_R M$ satisfies the ACC condition on $\mathcal{R}^c(M)$.*

3. *$\mathbf{Z}^c(M)$ is Artinian if and only if ${}_R M$ satisfies the DCC condition on $\mathcal{R}^c(M)$.*

Theorem 5.17. *Let ${}_R M$ be a top^c -module. If ${}_R M$ is Noetherian (Artinian), then $\mathbf{Z}^c(M)$ is Noetherian (Artinian).*

5.18. Recall that ${}_R M$ is said to be *distributive* iff

$$L \cap (K_1 + K_2) = (L \cap K_1) + (L \cap K_2) \text{ (equivalently, } (L + K_1) \cap (L + K_2) = L + (K_1 \cap K_2)) \quad (43)$$

for all submodules of M . We call ${}_R M$ *completely distributive* iff for all $L, K_\lambda \in \mathcal{L}(M)$:

$$\bigcap_{\lambda \in \Lambda} (L + K_\lambda) = L + \left(\bigcap_{\lambda \in \Lambda} K_\lambda \right).$$

Proposition 5.19. *Let ${}_R M$ be a completely distributive top^c -module and $\mathcal{A} \subseteq \text{Spec}^c(M)$.*

1. *If \mathcal{A} is irreducible, then $\mathcal{J}(\mathcal{A})$ is a coprime submodule of M .*

2. *If $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$, then the following are equivalent:*

- (a) $\mathcal{A} \subseteq \text{Spec}^c(M)$ is irreducible;
- (b) $\mathcal{J}(\mathcal{A})$ is a coprime submodule of M ;
- (c) $\mathcal{J}(\mathcal{A}) \leq_R M$ is strongly irreducible.

Proof. Let ${}_R M$ be a top^c -module and $\mathcal{A} \subseteq \text{Spec}^c(M)$.

1. Assume that \mathcal{A} is irreducible. By definition, $\mathcal{A} \neq \emptyset$ and so $\mathcal{J}(\mathcal{A}) \leq_R M$. Let $I \in \mathcal{I}(R)$ and suppose that $IM \not\subseteq \mathcal{J}(\mathcal{A})$ and $IM + \mathcal{J}(\mathcal{A}) \neq M$. Let $\mathcal{A}_1 := \{K \in \mathcal{A} \mid IM \subseteq K\}$ and $\mathcal{A}_2 := \{K \in \mathcal{A} \mid IM + K = M\}$. Notice that $\mathcal{A} \subseteq \mathcal{V}^c(\mathcal{J}(\mathcal{A}_1)) \cup \mathcal{V}^c(\mathcal{J}(\mathcal{A}_2))$. However, $\mathcal{A} \not\subseteq \mathcal{V}^c(\mathcal{J}(\mathcal{A}_1))$ (otherwise, $IM \subseteq \mathcal{J}(\mathcal{A}_1) = \mathcal{J}(\mathcal{A})$) and $\mathcal{A} \not\subseteq \mathcal{V}^c(\mathcal{J}(\mathcal{A}_2))$ (otherwise, $IM + \mathcal{J}(\mathcal{A}) = IM + \mathcal{J}(\mathcal{A}_2) = IM + \bigcap_{K \in \mathcal{A}_2} K = \bigcap_{K \in \mathcal{A}_2} (IM + K) = M$).

This is a contradiction, whence $IM \subseteq \mathcal{J}(\mathcal{A})$ or $IM + \mathcal{J}(\mathcal{A}) = M$. Consequently, $\mathcal{J}(\mathcal{A})$ is a coprime submodule of M .

2. (a) \Rightarrow (b) follows by “1”.

(b) \Rightarrow (c) is obvious.

(c) \Rightarrow (a) Since $\mathcal{J}(\mathcal{A})$ is a proper submodule of M , we conclude that $\mathcal{A} \neq \emptyset$. Suppose that $\mathcal{A} \subseteq \mathcal{V}^c(L_1) \cup \mathcal{V}^c(L_2) \subseteq \mathcal{V}^c(L_1 \cap L_2)$ for some $L_1, L_2 \leq_R M$, so that $L_2 \cap L_2 \subseteq \mathcal{J}(\mathcal{A})$. Since $\mathcal{J}(\mathcal{A}) \leq_R M$ is strongly irreducible, $L_1 \subseteq \mathcal{J}(\mathcal{A})$ so that $\mathcal{A} \subseteq \mathcal{V}^c(L_1)$ or $L_2 \subseteq \mathcal{J}(\mathcal{A})$ so that $\mathcal{A} \subseteq \mathcal{V}^c(L_2)$. We conclude that \mathcal{A} is irreducible. ■

Theorem 5.20. Let ${}_R M$ be a completely distributive top^c -module.

1. If $\text{Spec}^c(M)$ is irreducible, then $\text{Rad}_M^c(M)$ is a coprime submodule of M .
2. If $\text{Max}(M) \subseteq \text{Spec}^c(M)$ is irreducible, then $\text{Rad}_M^c(M)$ is a coprime submodule of M .
3. Let $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$.

(a) The following are equivalent:

- i. $\text{Spec}^c(M)$ is irreducible;
- ii. $\text{Rad}_M^c(M)$ is a coprime submodule of M ;
- iii. $\text{Rad}_M^c(M) \leq_R M$ is strongly irreducible.

(b) The following are equivalent:

- i. $\text{Max}(M) \subseteq \text{Spec}^c(M)$ is irreducible;
- ii. $\text{Rad}(M)$ is a coprime submodule of M ;
- iii. $\text{Rad}(M) \leq_R M$ is strongly irreducible.

Example 5.21. Let ${}_R M$ be a top^c -module. If $\emptyset \neq \mathcal{A} \subseteq \text{Spec}^c(M)$ is a chain, then \mathcal{A} is irreducible. In particular, if ${}_R M$ is uniserial, then $\text{Spec}^c(M)$ is irreducible.

Notation. We denote by $\text{Min}(\text{Spec}^c(M))$ the minimal elements of $\text{Spec}^c(M)$.

Proposition 5.22. *Let ${}_R M$ be completely distributive and $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$ (whence ${}_R M$ is a top^c -module).*

1. *The bijection (42) restricts to a bijection*

$$\text{Spec}^c(M) \longleftrightarrow \{\mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^c(M) \text{ is an irreducible closed subset}\}.$$

2. *The bijection (42) restricts to a bijection*

$$\text{Min}(\text{Spec}^c(M)) \longleftrightarrow \{\mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^c(M) \text{ is an irreducible component}\}.$$

Proof. Recall the bijection $\mathcal{R}^c(M) \xrightarrow{\mathcal{V}^c(-)} \mathbf{CL}(\mathbf{Z}^c(M))$.

1. Let $K \in \text{Spec}^c(M)$. Then $K = \mathcal{J}(\mathcal{V}^c(K))$ and it follows that the closed set $\mathcal{V}^c(K)$ is irreducible by Proposition 5.20. On the other hand, let $\mathcal{A} \subseteq \text{Spec}^c(M)$ be a closed irreducible subset. Then $\mathcal{A} = \mathcal{V}^c(L)$ for some $L \leq_R M$. Notice that $\mathcal{J}(\mathcal{A})$ is coprime in M by Proposition 5.20 and that $\mathcal{A} = \overline{\mathcal{A}} = \mathcal{V}^c(\mathcal{J}(\mathcal{A}))$.
2. Let K be minimal in $\text{Spec}^c(M)$. Then $\mathcal{V}^c(K)$ is an irreducible subset of $\text{Spec}^c(M)$ by “1”. By [Bou1998], $\mathcal{V}^c(K)$ is contained in some irreducible component \mathcal{Y} of $\text{Spec}^c(M)$. Since \mathcal{Y} is closed, there exists by “1” some $L \in \text{Spec}^c(M)$ such that $\mathcal{Y} = \mathcal{V}^c(L)$. If $\mathcal{V}^c(K) \subsetneq \mathcal{V}^c(L)$, then $L \subsetneq K$, a contradiction. Consequently, $\mathcal{V}^c(K) = \mathcal{V}^c(L)$ is an irreducible component of L .

On the other hand, let \mathcal{Y} be an irreducible component of $\text{Spec}^c(M)$. Then \mathcal{Y} is closed and irreducible, i.e. $\mathcal{Y} = \mathcal{V}^c(L)$ for some $L \in \text{Spec}^c(M)$ by “1”. Suppose that L is not minimal in $\text{Spec}^c(M)$, so that there exists $K \in \text{Spec}^c(M)$ such that $K \subsetneq L$. It follows that $\mathcal{V}^c(L) \subsetneq \mathcal{V}^c(K)$, a contradiction since $\mathcal{V}^c(K) \subseteq \text{Spec}^c(M)$ is irreducible by “1”. We conclude that L is minimal in $\text{Spec}^c(M)$. ■

Corollary 5.23. *If ${}_R M$ is completely distributive and $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$, then $\text{Spec}^c(M)$ is a Sober space.*

Theorem 5.24. *Let ${}_R M$ be a coatomic top^c -module. Then ${}_R M$ is hollow if and only if $\text{Spec}^c(M)$ is ultraconnected.*

Example 5.25. Let ${}_R M$ be a coatomic multiplication module. Then $\text{Spec}^c(M)$ is ultraconnected if and only if ${}_R M$ is local. Indeed, if $\text{Spec}^c(M) = \text{Max}(M)$ is ultraconnected, then $|\text{Max}(M)| = 1$: If $\mathfrak{m}_1, \mathfrak{m}_2 \in \text{Max}(M)$ are distinct, then $\mathcal{V}(\mathfrak{m}_1) \cap \mathcal{V}(\mathfrak{m}_2) = \{\mathfrak{m}_1\} \cap \{\mathfrak{m}_2\} = \emptyset$, a contradiction. Since ${}_R M$ is coatomic, we conclude that ${}_R M$ is local. On the other hand, if ${}_R M$ is local, then indeed ${}_R M$ is hollow whence $\text{Spec}^c(M)$ is ultraconnected by Theorem 5.24. ■

Theorem 5.26. *Let ${}_R M$ be a coatomic top^c -module.*

1. *If $\text{Max}(M)$ is countable, then $\text{Spec}^c(M)$ is countably compact.*
2. *If $\text{Max}(M)$ is finite, then $\text{Spec}^c(M)$ is compact.*

Example 5.27. Let $M = \mathbb{Z}_{p^\infty}$ and recall that

$$\text{Spec}^c(\mathbb{Z}_{p^\infty}) = \{\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}) \mid n \in \mathbb{N}\}.$$

Notice that $\text{Max}(M) = \emptyset$ (in particular finite) but $\text{Spec}^c(\mathbb{Z}_{p^\infty})$ is not compact since the open cover

$$\{\mathcal{X}^c\left(\mathbb{Z}(\frac{1}{p^{n+1}} + \mathbb{Z})\right) \mid n \in \mathbb{N}\}$$

has no finite subcover. Recall that \mathbb{Z}_{p^∞} is not coatomic as a \mathbb{Z} -module. This shows that the assumption that ${}_R M$ be a coatomic in Theorem 5.26 “2” cannot be removed.

Proposition 5.28. *Let ${}_R M$ be a top^c -module and assume that every coprime submodule of M is maximal.*

1. *If ${}_R M$ has the complete max-property, then $\text{Spec}^c(M)$ is discrete.*
2. *M has a unique maximal R -submodule if and only if ${}_R M$ has the complete max-property and $\text{Spec}^c(M)$ is connected.*

Theorem 5.29. *Let ${}_R M$ be a coatomic top^c -module and assume that every coprime submodule of M is maximal. If ${}_R M$ has the complete max property, then*

1. *$\text{Spec}^c(M)$ is countably compact if and only if $\text{Max}(M)$ is countable.*
2. *$\text{Spec}^c(M)$ is compact if and only if $\text{Max}(M)$ is finite.*

As a direct consequence of Proposition 5.28 we obtain:

Theorem 5.30. *Let ${}_R M$ be a coatomic top^c -module with the complete max-property and assume that every coprime submodule of M is maximal. Then ${}_R M$ is local if and only if $\text{Spec}^c(M)$ is connected.*

Lemma 5.31. *Let $\text{Spec}^c(M) \subseteq \mathcal{SI}(M)$. If $n \geq 2$ and $\mathcal{A} = \{K_1, \dots, K_n\} \subseteq \text{Spec}^c(M)$ is a connected subset, then for every $i \in \{1, \dots, n\}$, there exists $j \in \{1, \dots, n\} \setminus \{i\}$ such that $K_i \leq_R K_j$ or $K_j \leq_R K_i$.*

Proposition 5.32. *Let ${}_R M$ be a coatomic top^c -module with the complete max-property and let $\emptyset \neq \mathcal{A} = \{K_\lambda\}_\Lambda \subseteq \text{Max}(M)$. If $|\mathcal{M}(L)| < \infty$ for every $L \in \text{Spec}^c(M)$, then \mathcal{A} is locally finite.*

Proposition 5.33. *If ${}_R M$ be a coatomic top^c -module, then the following are equivalent for any $L \leq_R M$:*

1. $L \in \text{Max}(M)$;
2. L is a coprime submodule of M and $\mathcal{V}^c(L) = \{L\}$;
3. $\{L\}$ is closed in \mathbf{Z}_M^c .

Proposition 5.34. *Let ${}_R M$ be a coatomic top^c -module. Then $\text{Spec}^c(M) = \text{Max}(M)$ if and only if $\mathbf{Z}^c(M)$ is T_1 (Frécht space).*

Combining the previous results we obtain

Theorem 5.35. *Let ${}_R M$ be a coatomic top^c -module with the complete max-property. The following are equivalent:*

1. $\text{Spec}^c(M) = \mathcal{S}(M)$;
2. $\mathbf{Z}^c(M)$ is discrete;
3. $\mathbf{Z}^c(M)$ is T_2 (Hausdorff space);
4. $\mathbf{Z}^c(M)$ is T_1 (Frécht space).

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